Modeling the Term Structure of Defaultable Bonds under Recovery Risk

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Abstract

Although the Basle Committee has identified recovery risk as an important source of risk in addition to default, the impact of recovery rates on bond prices is not yet fully understood in the literature. This paper proposes a reduced form discrete-time approach for pricing defaultable securities incorporating stochastic recovery rates. We provide pricing formulas for risky bonds and CDS contracts in the case of an economy with an affine state vector. We show that our methodology provides a tractability that is usually typical of continuous time models. The model stays tractable even in the case of rich and realistic econometric representations of the state variables and can therefore be estimated using standard techniques. As an illustration, several specifications for the state variables are discussed and a numerical example is provided.

Keywords: defaultable bonds, CDS contracts, correlation, stochastic recovery rates, discrete time.

JEL Classification: G12, G13

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1 Introduction

The credit risk literature contains two types of approaches for modeling the term structure of defaultable bonds: *structural models* and *reduced form* models. In structural models, credit events are triggered by changes in the firm's value relative to some barrier. This approach was originally proposed by Merton (1974) who assumes that default occurs once the firm exhausts all its assets.[1] However, the structural approach is not straightforward to implement because generally the firm's assets are neither observable nor tradeable. In the reduced form approach, the value of the firm and its capital structure are not explicitly modeled, and the credit events are specified in terms of a jump process. This approach to modeling default risk considers the default event as an unpredictable stopping time. Existing reduced form models (see Jarrow, Lando and Turnbull (1997), Lando (1998), Duffie and Singleton (1999)) use a continuous time framework and show that the price of any defaultable contingent claim can be computed as a conditional mathematical expectation with a modified discount factor. This allows standard risk-free term structure tools to be used for both theoretical and empirical purposes. It also makes reduced form models particularly suited for empirical implementation in both bonds and Credit Default Swaps markets.

The aim of this paper is to develop a discrete time reduced form approach for pricing defaultable bonds incorporating a stochastic risk free interest rate, default intensity and recovery rate. Contingent claim valuation in discrete time economies was previously studied by Rubinstein (1976) and Brennan (1976). Turnbull and Milne (1991) extend this approach to the case of an economy where interest rates are stochastic and derive closed form solutions for several interest rate derivatives. In the credit risk literature, Das and Sundaram (2000) adopt a discrete time reduced form model. Their approach is based on the lattice version of the Heath, Jarrow and Morton model (1990) and allows for pricing credit derivatives under the assumption of recovery of market value with constant recovery rate. Our approach is not restricted to a particular recovery assumption and distinguishes itself from Das and Sundaram (2000) in that the state space is infinite. The framework developed in this paper is tractable and allows us to capture two important stylized facts characterizing the term structure of defaultable bonds. The first stylized fact is the time varying nature of the recovery rate. Empirical evidence, including Altman (2002), suggests that recovery rates exhibit substantial variability. Acharya, Bharath and Srinivasan (2004) show also that the recovery rate is lower in a distressed economy than in a healthy economy. These findings prove that adequate modeling of the term structure of risky bonds should take into account the stochastic nature of the recovery rate. The second stylized fact is the correlation between recovery rates, the risk free term structure and default risk. While most of literature focuses on the link between default risk and the risk free term structure (See Duffee (1998)), the impact of recovery rates on the term structure of credit spreads has largely been ignored. This paper fills this void by modeling the recovery rate in addition to the pricing kernel and the hazard rate, as well as their correlations. In most of the existing literature, allowing for stochastic recovery rates is relatively costly in terms of the computational burden. Computing the value of a risky bond with a stochastic recovery rate generally calls for one or more numerical integra-

[1] In Merton's model, default can only occur at maturity. Black and Cox (1976) relax this assumption by allowing default to occur when the firm's value reaches a lower barrier. Later, many papers extended the work of Merton (1974) and Black and Cox (1976), among them: Leland (1994), Leland and Toft (1996) and Collin-Dufresne and Goldstein (2001).

tions. The literature has therefore made simplifying assumptions. Jarrow, Lando and Turnbull (1997) assume that the recovery rate is constant. Das and Tufano (1995) model the time varying nature of the recovery rate and adopt a discrete time economy where the spread is decomposed into two components: the default risk and the stochastic recovery rate. They assume that the default risk is independent of the risk free term structure to keep the model tractable. One notable exception is the recent paper of Bakshi, Madan and Zhang (2002) who explicitly model a stochastic recovery rate and derive pricing solutions for prices of risky bonds. These pricing solutions are based on the characteristic function of the state variables. Nevertheless, numerical integration has to be used even though the characteristic function of the state vector is available analytically. In a multifactor set up, the solution is complex and computationally intensive, a drawback that constitutes an obstacle for an empirical implementation.

As an alternative, we propose a general equilibrium framework for pricing risky bonds in which we adopt a finite horizon representative agent economy with discrete time trading. In this economy, equilibrium is characterized via the pricing kernel (see Duffie (2001) for a general presentation) and default time corresponds to the first jump time of a Cox Process (see Lando (1998)). We model the pricing kernel, the integrated hazard rate and the recovery rate as discrete time adapted processes and show that a discrete time methodology is not only able to retain much of the intuition underlying the continuous time valuation framework but is also more tractable than existing models. More precisely, pricing formulas are derived for defaultable bonds and credit default swaps (CDS) contracts when the recovery rate is stochastic. We also examine different recovery assumptions for risky bonds: recovery of Treasury (RT), recovery of face value (RFV) and recovery of market value (RMV). Provided that the conditional Laplace transform of the state vector is analytically known, our approach admits closed form solutions for prices of CDS contracts and risky bonds under RT and RFV. These solutions hold for a large class of affine discrete time dynamics and are easy to compute since they only involve a finite summation. Under the assumption of RMV, Monte Carlo simulation has to be used for pricing the defaultable bond, but our model allows us to empirically disentangle variations of recovery and hazard rates, unlike the continuous-time RMV case. The affine family contains a large variety of dynamics including the Gaussian AR(1) and the Markov Gamma process. We provide two illustrative examples using these dynamics in order to demonstrate the analytical tractability of the model and the implications of stochastic recovery rates.

The rest of the paper is structured as follows. In section 2, we discuss the tractability of reduced form models in the presence of recovery risk. In section 3, we present a general framework for pricing bonds and CDS contracts under recovery risk. In section 4, we explore the pricing solutions for the family of affine processes and prove that our methodology provides a tractability that is usually typical of continuous time models. In section 4, we offer some concluding remarks. Proofs are collected in an Appendix.

2 Analytical Tractability of Reduced-Form Models

The choice between continuous time and discrete time models is often a question of tractability. In many applications, continuous time models offer a very attractive valuation framework because closed form solutions are generally available. However, when used in a context of stochastic recovery rates, continuous time models yield pricing solutions that are often complex. To motivate the discrete time methodology of this paper, this section summarizes the continuous time framework of Lando (1998) and Duffie and Singleton (1999). In particular, we show that pricing bonds under recovery risk in a fairly rich econometric specification is computationally intensive and thus quite difficult to be empirically implemented.

2.1 Pricing Defaultable Bonds in Continuous Time

Let the process $\{r_t, t \geq 0\}$ represent the risk free short rate and $\{\lambda_t, t \geq 0\}$ be the hazard rate of the default time. Assuming that the risky bond has a maturity denoted by T, a face value equal to \$1, and a random recovery payment at the time of default of the form Z_τ, Lando (1998) shows that the price of a zero coupon bond can be computed as follows

$$\overline{B}(t,T) = E_t^Q \left[\exp\left(-\int_t^T (r_s + \lambda_s)\, ds \right) \right] + E_t^Q \left[\int_t^T Z_s \lambda_s \exp\left(-\int_t^s (r_u + \lambda_u)\, du \right) ds \right], \quad (2.1)$$

where Q denotes the Equivalent Martingale Measure.

The first term accounts for the promised face value and the second conditional expectation reflects the recovery value that the bondholder receives upon default. The literature contains three different assumptions to model the recovery payment: recovery of face value (RFV) (see Brennan and Schwartz (1980), Duffee (1999), Jarrow, Lando and Turnbull (1997) and Lando (1998)), recovery of treasury (RT) (see Longstaff and Schwartz (1995) and Jarrow and Turnbull (1995)) and recovery of market value (RMV) (Duffie and Singleton (1999)).

Under the RFV assumption, the recovery payment is a fraction of the face value

$$Z_s = (1 - L_s) \quad (2.2)$$

where $\{L_s, s > t\}$ is an adapted process bounded by 1. This process corresponds to the loss given default, whereas $\{(1 - L_s), s > t\}$ corresponds to the recovery process.

Under the RT assumption, the recovery payment is a fraction of a Treasury bond with maturity T and face value equal to \$1

$$Z_s = (1 - L_s) \times B(s,T) \quad (2.3)$$

Under the RMV assumption, Z_s is given by

$$Z_s = (1 - L_s) \times \overline{B}(s^-, T) \quad (2.4)$$

where $\overline{B}(s^-, T)$ denotes the price of the risky bond just before default.

The choice between these recovery assumptions depends on the legal structure of the instrument to be priced. If, for instance, one assumes liquidation at default and the absolute priority rule applies, then the recovery of face value or the recovery of treasury value are more realistic. However, if liquidation is avoided, then one can use the recovery of market value assumption. More generally, several credit events such as bankruptcy, repudiation and restructuring affect the value of defaultable securities with different probabilities and thus result in different recovery rates. As pointed out by Duffie and Singleton (1999), from a pricing viewpoint, the choice between different recovery assumptions in continuous time may also be potentially motivated by computational constraints. In general, r_s, λ_s and L_s are all functions of some state variables in the economy. A key ingredient in computing $\overline{B}(t,T)$ is the conditional joint distribution of the random vector (r_s, λ_s, L_s). Analytical expressions are generally not available unless one makes specific assumptions about the dynamics of (r_s, λ_s, L_s). For this reason, simplifying assumptions have been made in the literature, such as independence between the short rate and the hazard rate processes as in Jarrow and Turnbull (1995), or a constant recovery process as in Jarrow, Lando and Turnbull (1997) and Duffee (1998).

In addition to their computational tractability, an important feature that differentiates these recovery assumptions is their ability to identify the impact of the hazard and the recovery rate. While the RFV and RT allow for identification of the impact of the recovery rate, this is not the case under the RMV assumption. Recall that under the RMV assumption, equation (2.1) is equivalent to a recursive stochastic integral equation whose solution is, for a zero-coupon risky bond

$$\overline{B}^{RMV}(t,T) = E_t^Q\left[\exp\left(-\int_t^T (r_s + L_s\lambda_s)\,ds\right)\right]. \tag{2.5}$$

The RMV assumption is very convenient for pricing credit derivatives, since it only relies on the explicit modeling of the short rate and the so-called “mean loss rate” $(L_s\lambda_s)$. However, $\{L_s, s > t\}$ cannot be allowed to vary stochastically, otherwise it becomes impossible to distinguish between variations in the hazard rate and variations in the recovery rates, as explicitly recognized by Duffie and Singleton (1999). As a consequence, although the RMV assumption has become the standard assumption in credit risk modeling (see Duffie and Singleton (1997) and Duffee (1999)), the limited literature that explicitly models the recovery rate uses the RT and RFV assumptions.

A recent paper by Bakshi, Madan and Zhang (2002) develops a general pricing solution under these assumptions. Using the characteristic function of the state variables combined with the pricing equation (2.1), Bakshi, Madan and Zhang (2002) derive the general pricing solutions under the RT and RFV assumptions when the recovery rate is stochastic. Nevertheless, closed form expressions for prices of risky bonds are still difficult to compute even when the characteristic function is available analytically. The use of numerical techniques such as the Gauss-Laguerre quadrature is required and the computational cost is significant even in the case of a single factor model. Mainly for this reason, Bakshi, Madan and Zhang (2002) model the recovery rate as a function of the hazard rate. Following empirical evidence in Altman (2001) that suggests an inverse relation between recovery rate and default risk, they parametrize the hazard rate as an affine function of the short rate and the recovery rate as inversely exponentially related to the hazard rate

$$(1 - L_s) = \omega_0 + \omega_1 \exp(-\lambda_s), \tag{2.6}$$

$$\lambda_s = \gamma_0 + \gamma_1 r_s \text{ and } dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}dW_t. \tag{2.7}$$

This assumption allows for closed form expressions that are rather complex to evaluate even for the case of single factor. For instance, the price of the risk bond under the assumption of RT is

$$\begin{aligned}\overline{B}^{RT}(t,T) &= \exp(-\gamma_0(T-t)) \times G\left(t,T,i(1+\gamma_1),0\right) \\ &+\omega_0\gamma_0 \int_t^T \exp(-\gamma_0(u-t)) \times G\left(t,u,i(1+\gamma_1),0\right) \times G(t,u,i,0)du \\ &-i\omega_0\gamma_1 \int_t^T \exp(-\gamma_0(u-t)) \times G_\nu\left(t,u,i(1+\gamma_1),0\right) \times G(t,u,i,0)du \\ &+\omega_1\gamma_0 \exp(-\gamma_0) \int_t^T \exp(-\gamma_0(u-t)) \times G\left(t,u,i(1+\gamma_1),i\gamma_1\right) \times G(t,u,i,0)du \\ &-i\omega_1\gamma_1 \exp(-\gamma_0) \int_t^T \exp(-\gamma_0(u-t)) \times G_\nu\left(t,u,i(1+\gamma_1),i\gamma_1\right) \times G(t,u,i,0)du,\end{aligned} \tag{2.8}$$

where $G(t,u,\phi,\upsilon)$ denotes the characteristic function of the pair $\left(\int_t^u r_s ds, r_u\right)$

$$G(t,u,\phi,\upsilon) = E_t^Q\left[\exp(i\phi\int_t^u r_s ds + \upsilon r_u)\right]. \tag{2.9}$$

This example illustrates the computational burden that results from an explicit modeling of stochastic recovery rates and the trade-off between the econometric richness of the model and its computational tractability. It is indeed well known in the literature that a single factor model is unable to fit the risk free term structure[2]. A multifactor extension is therefore necessary in order to capture all the stylized facts of the term structure. Such an extension would, however, result in a more complex solution that requires the numerical integration of additional terms. An empirical implementation of a multifactor model is therefore quite difficult. More generally, although the set of models for which valuation results are available in the existing literature is quite large (see Duffie, Pan and Singleton (2000)), the lack of tractability of these models is often an obstacle for their implementation. In the next section, we illustrate the trade-off between the econometric richness and the analytical tractability of continuous time models in the general case of an affine state vector.

2.2 Pricing solutions in Affine Models

Let us assume that the economy is driven by an affine state vector Y_t (see Duffie, Pan and Singleton (2000)). The conditional probability distribution of the state vector is characterized via the Laplace and the extended Laplace transforms of the pair $\left(\int_t^T Y_s ds, Y_T\right)$. Under some technical conditions, the Laplace transform of this pair is established as follows

$$\begin{aligned}G(t,T,u,v) &= E_t^Q\left[\exp\left(u\int_t^T R_s ds + vY_T\right)\right] \\ &= \exp\left(\alpha\left(t,T,u,v\right) + \beta\left(t,T,u,v\right)Y_t\right),\end{aligned} \tag{2.10}$$

[2] Litterman and Scheinkman (1991) find that 3 factors are needed to explain at least 95% of the variation in yields.

where $R_t = a_0 + b_0 Y_t$ and $(u, v) \in (\mathbb{R}^N, \mathbb{R}^N)$. α and β satisfy a set of $(N+1)$ ordinary differential equations that can be solved numerically.

The extended Laplace transform is

$$\begin{aligned} J(t,T,u,v,w) &= E_t^Q \left[\exp\left(u \int_t^T R_s ds + vY_T \right) (wY_T) \right] \\ &= G(t,T,u,v) \left(A(t,T,u,v,w) + B(t,T,u,v,w) Y_t \right), \end{aligned} \tag{2.11}$$

where A and B satisfy an other set of $(N+1)$ ordinary differential equations that are in general solved numerically.

Furthermore, we assume that N is equal to 5 and specify the short rate, the hazard rate and the recovery rate as follows

$$r_t = \delta_0 + \sum_{i=1}^{3} \delta_{1i} Y_{it}, \tag{2.12}$$

$$\lambda_t = \gamma_0 + \sum_{i=1}^{4} \gamma_{1i} Y_{it} \text{ and} \tag{2.13}$$

$$(1 - L_s) = \exp(-\phi Y_t) \tag{2.14}$$

In order to guarantee that the recovery rate is bounded by one, the coefficients ϕ_{1i} are constrained to be positive. Consistent with the recent empirical literature (Dai and Singleton (2000), Duffee (2002)), a three factor model is postulated for the risk-free term structure. The hazard rate contains an additional factor that can be thougth as idiosyncratic and the recovery rate involves a fifth factor that reflects some firm-specific behavior. In order to illustrate the complexity of the pricing solutions in affine models, we compute the price of the risky bond under the assumption of RFV. Under some technical conditions, this latter is

$$\overline{B}(t,T) = G(t,T,u_1,0) \times F + \int_t^T G(t,s,u,\phi) \left(A(t,s,u_1,\phi,w_1) + B(t,s,u_1,\phi,w_1) Y_t + \gamma_0 \right) ds \times F, \tag{2.15}$$

where

$$u_1 = - \left[\begin{array}{ccccc} (\delta_{11} + \gamma_{11}) & (\delta_{12} + \gamma_{12}) & (\delta_{13} + \gamma_{13}) & \gamma_{14} & 0 \end{array} \right], \tag{2.16}$$

and

$$w_1 = - \left[\begin{array}{ccccc} \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & 0 \end{array} \right].$$

For a given maturity, the first term is available in closed form up to the solution of a set of 6 ODE's. However, the second term involves the computation of 6 integrals and each of these integrals relies on the numerical solutions of a set of 12 ODE's. The frequent use of numerical approximation techniques substantially increases the computational complexity and makes the model not flexible enough for a practical use.

At this point, it would be interesting to see if the availability of a closed form solution of the Laplace transform reduces the computational complexity and improves the tractability of the model. This is in deed the case for several affine diffusions, including the correlated square-root process and the multifactor Gaussian process. Intuitively, if the Laplace transform is available in closed form and thus does not require the numerical solution of ODE's, one would expect the model to gain in tractability. As will be clear in this example, such is not the case and it turns out that the computational burden still prevails even when the Laplace transform is available in closed form.

Suppose that $\alpha(t,T,u,v)$ and $\beta(t,T,u,v)$ are available in closed form. The extended Laplace transform can then be expressed as

$$\begin{aligned} J(t,T,u,v,w) &= E_t^Q\left[\exp\left(u\int_t^T R_s ds + vY_T\right)(wY_T)\right] \\ &= \sum_{i=1}^N w_i \times \frac{\partial G(t,T,u,v)}{\partial v_i} \\ &= \sum_{i=1}^N w_i \times \left(\frac{\partial \alpha(t,T,u,v)}{\partial v_i} + \sum_{i=1}^N \frac{\partial \beta_i(t,T,u,v)}{\partial v_i} Y_{it}\right). \\ &\quad \times \exp\left(\alpha(t,T,u,v) + \beta(t,T,u,v) Y_t\right). \end{aligned} \tag{2.17}$$

In the five-factor model described earlier, the price of the risky bond under the assumption of RFV is

$$\begin{aligned} \overline{B}(t,T) &= G(t,T,u_1,0)\times F + \sum_{j=1}^5 \int_t^T w_j \times \left(\frac{\partial \alpha(t,s,u_1,\phi)}{\partial v_i} + \sum_{i=1}^5 \frac{\partial \beta_i(t,s,u_1,\phi)}{\partial v_i} Y_{it}\right) \\ &\quad \times \exp\left(\alpha(t,s,u_1,\phi) + \beta(t,s,u_1,\phi) Y_t\right) ds \times F \\ &\quad + \int_t^T \exp\left(\alpha(t,T,u_1,\phi) + \beta(t,T,u_1,\phi) Y_t\right)\gamma_0. \end{aligned} \tag{2.18}$$

This pricing solution involves the numerical evaluation of 31 integrals which definitely proves that the computational cost is still significant even though the Laplace transform is known analytically. It also underlines the importance of building a tractable model for pricing defaultable securities under recovery risk.

In the next section, we introduce our model and derive general pricing solutions for bonds and CDS contracts under recovery risk.

3 A Framework for Pricing Defaultable Securities under Recovery Risk

3.1 The Financial Market

We assume an economy with a frictionless financial market where trading follows a discrete time sequence $t, t+1, ..., t+n$. In this economy, two classes of zero-coupon bonds are traded: risk free bonds and defaultable bonds. The uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The default probability is modeled as in reduced form models but with a slight modification. We assume that default can occur any time between two consecutive trading dates and that its time, τ, corresponds to the first time jump of a Cox process with hazard rate $\{\lambda(s), 0 \le s \le T\}$. The probabilistic framework has the particularity that the filtration $\{\mathcal{F}_{t+i}, i = 0, ..., n\}$ contains the information on whether default occurred before $t+i$ in addition to the information on the risk free term structure and any other relevant state variables. If we denote by $\{\mathcal{G}_{t+i}, i=0,...,n\}$ the basic filtration without the information on the occurrence of default, then $\mathcal{F}_{t+i}$ can be written as

$$\mathcal{F}_{t+i} = \mathcal{G}_{t+i} \vee \sigma\left(\tau < s, s \leq t+i\right),$$

where the sigma field $\sigma\left(\tau < s, s \leq t+i\right)$ holds the information on whether default occurred before $t+i$.

Under technical conditions (see Lando (1998)), the probability of no jump is then defined as

$$\begin{aligned} \Pr\left(\tau > t+k \mid \mathcal{F}_t\right) &= E\left[\exp(-\int_t^{t+k} \lambda_s ds) \mid \mathcal{G}_t\right] && (3.1) \\ &= E_t\left[\exp(-\sum_{i=1}^{k} \Lambda_{t+i})\right], && (3.2) \end{aligned}$$

where the subscript t denotes the information contained in the basic filtration $\{\mathcal{G}_{t+i}, i=0,...,n\}$[3] and

$$\Lambda_{t+i} = \int_{t+i-1}^{t+i} \lambda_s ds < \infty. \tag{3.3}$$

In this paper, rather than modeling $\lambda\left(u\right)$, we model $\{\Lambda_{t+i}, i=1,...,n\}$ as a discrete time process. This assumption is actually equivalent to a piecewise constant modeling of the hazard rate.

For pricing defaultable securities, we need the information on whether default occurred prior to any trading date $t+i$ and thus conditioning on $\mathcal{F}_{t+i}$ is necessary. For any asset with price X_t and dividend payment D_t, the following Euler equation has to be satisfied at time t

$$X_t = E\left[M_{t,t+1}\left(X_{t+1}+D_{t+1}\right) \mid \mathcal{F}_t\right]. \tag{3.4}$$

where $\left\{M_{t+i-1,t+i} = \frac{U'(c_t)}{U'(c_{t+1})}, i=1,...,n\right\}$ is a discrete time $\mathcal{G}_{t+i}$ measurable process that corresponds to the pricing kernel between $t+i-1$ and $t+i$.

Modeling the pricing kernel is equivalent to specify the risk free term structure. Using equation (3.4), the price at time t of a risk free zero-coupon bond with maturity $t+n$ is

$$\begin{aligned} B(t,t+n) &= E\left[\prod_{i=1}^{n} M_{t+i-1,t+i} \mid \mathcal{G}_t\right] && (3.5) \\ &= E_t\left[M_{t,t+n}\right]. \end{aligned}$$

We now turn to the third component of the term structure of defaultable bonds: the recovery rate. Our model distinguishes itself from continuous time models in that default can occur at any time between two consecutive trading dates and the recovery payment is received at the next trading date following the default time. This hypothesis is fundamental and makes this framework very attractive for pricing CDS contracts and other credit derivatives. It also provides the flexibility that is needed to obtain tractable closed form solutions for prices of risky

[3] Throughout this paper, the subscript t is equivalent to a conditioning on the filtration $\mathcal{G}_t$.

bonds. For simplicity, we consider a defaultable security that yields \$1 at $t+n$ if no default has occurred. Upon default, the security's payment (at the next trading date) is described by an adapted stochastic process $\mathcal{Z}=\{Z_{t+i}, i=1,...n\}$.

In the next two sections, we assemble all these ingredients and provide general pricing solutions for bonds and CDS contracts under recovery risk.

3.2 Pricing Bonds under Recovery Risk

The price of a zero-coupon risky bond with a recovery payment at the next trading date following the default time is obtained via the Euler equation (3.4). The value of the risky bond, $\overline{B}(t,n)$, verifies the following recursive equation

$$\overline{B}(t,t+n)=E\left[M_{t,t+1}\left[\overline{B}(t+1,t+n)\mathbb{I}_{(\tau>t+1)}+Z_{t+1}\mathbb{I}_{(\tau<t+1)}\right]\mid\mathcal{F}_t\right], \tag{3.6}$$

where the boundary condition is

$$\overline{B}(t+n-1,t+n)=E\left[M_{t+n-1,t+n}\left[\mathbb{I}_{(\tau>t+n)}+Z_{t+n}\mathbb{I}_{(\tau<t+n)}\right]\mid\mathcal{F}_{t+n-1}\right]. \tag{3.7}$$

Under the RFV assumption, the recovery payment is

$$Z_{t+i}=(1-L_{t+i}), \tag{3.8}$$

where $\{L_{t+i}, i=0,1,....,n\}$ is a discrete time $\mathcal{G}_{t+i}$ adapted process that is bounded by 1.

Under the RT assumption, the recovery payment is

$$Z_{t+i}=(1-L_{t+i})\,B(t+i,t+n), \tag{3.9}$$

where $B(t+i,n)$ is the price at $t+1$ of a Treasury bond with maturity $t+n$.

Under the RMV assumption, the recovery payment is

$$Z_{t+i}=(1-L_{t+i})\,\overline{B}^{RMV}(t+i,t+n), \tag{3.10}$$

where, $\overline{B}^{RMV}(t+i,n)$ denotes the price of the defaultable bond under the assumption of recovery of market value.

Solving equations (3.6) and (3.7) recursively yields the price of the risky bond under each of these assumptions. The following proposition synthesizes the pricing solutions for risky bonds under the three recovery assumptions described earlier.

Proposition 1 *The price at t of a risky bond with maturity $t+n$ has to satisfy the following general pricing solutions:*

1. Under the assumption of RT:

$$\begin{aligned}\overline{B}^{RT}(t,t+n) &= E_t\left[M_{t,t+n}\right]-\Bigg(E_t\left[L_{t+1}M_{t,t+n}\right]-E_t\left[M_{t,t+n}L_{t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right]\\ &\quad -E_t\left[M_{t,t+n}\left(\sum_{p=1}^{n-1}\exp(-\sum_{i=1}^{p}\Lambda_{t+i})\left(L_{t+p}-L_{t+p+1}\right)\right)\right]\Bigg).\end{aligned} \tag{3.11}$$

2. *Under the assumption of RFV:*

$$\begin{aligned}\overline{B}^{RFV}(t,t+n) &= E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right] \\ &+E_t\left[M_{t,t+1}\left(1-\exp(-\Lambda_{t+1})\right)\left(1-L_{t+1}\right)\right] \\ &+\sum_{p=2}^{n}E_t\left[M_{t,t+p}\exp\left(-\sum_{i=1}^{p-1}\Lambda_{t+i}\right)\left(1-\exp(-\Lambda_{t+p})\right)\left(1-L_{t+p}\right)\right].\end{aligned} \tag{3.12}$$

3. *Under the assumption of RMV:*

$$\overline{B}^{RMV}(t,t+n) = E_t\left[\prod_{i=1}^{n}M_{t+i-1,t+i}\left(1-L_{t+i}\left(1-\exp(-\Lambda_{t+i})\right)\right)\right]. \tag{3.13}$$

The subscript t denotes the current information relevant in determining values of state variables.
Proof. *See Appendix.* ■

Intuitively, this Proposition states that the value of the risky bond is a linear combination of expectations of discounted cash flows where the discount factor includes the intensity of default and the recovery rate in addition to the pricing kernel. Although these pricing solutions are established in their most general form, one can already draw some preliminary conclusions. Under the assumption of RT, the price of the defaultable bond is equal to the price of the risk free bond minus a discount factor that accounts for both default and recovery risk. Under the assumption of RFV, the price of the risky bond is equal to price of a risky bond that has zero recovery in the case of default in addition to terms that compensates for the existence of a positive recovery in the case of default. Under the assumption of RMV, the price of the risky bond is the discrete time equivalent of the Duffie and Singleton (1999) formula. It is worth emphasizing, however, that fixing the Loss Given Default in the discrete time framework is not necessary to achieve identification. Unlike the continuous time case, the Loss Given Default and the hazard rate do not enter the pricing equation symmetrically which alows the Loss Given Default to be identified and the assumption of RMV to be empirically investigated.

Obviously, a key ingredient in using these general solutions is the specification of the conditional joint distribution of the state variables. Once the conditional distribution of the state variable is characterized, the price of the risky bond is computed using (3.11) , (3.12) and (3.13). Adequate modeling of the term structure of credit spread requires specifying the relationship between these state variables in a way that allows the model to be consistent with some stylized facts. More precisely, the model has to be able to provide some new insights about the relationship between the three components of the term structure of defaultable bonds: the risk free term structure, the default risk and the recovery rate. Moreover, the model should also provide a suitable framework for pricing credit derivatives. In the next section, we show how this model can be applied to the valuation of one of the most popular credit derivatives: a CDS contract. Once again, we provide pricing formula in the case of a stochastic recovery rate.

3.3 Pricing CDS Contracts under Recovery Risk

Dealing with default and recovery risk often implies the use of financial securities that offer a protection against these sources of risk. A popular instrument that offers such a protection is the

CDS contract. The mechanism of a CDS contract works as follows: Consider two companies 'A' (the buyer) and 'B' (the seller) who enter into a contract that terminates at the time of a credit event or at a specified maturity, whichever occurs first. A credit event could be, for instance, a default of a third company 'C', called the reference company. It also includes other events such as bankruptcy, downgrade, failure to pay, repudiation or restructuring of the reference company. If a credit event occurs before the specified maturity, then company 'B' pays company 'A' a certain compensation in the form of a cash amount. This compensation is a sort of protection for company 'A' against a credit event of the reference company. Typically, the buyer (company 'A') has bought a defaultable bond from the reference company and is expecting future payments that correspond to the coupons and the face value. If the credit event is specified to be a default, the contract is called a default swap and it terminates by the default of the reference company. In such a contract, company 'B' offers a protection against the risk of default of the reference company and the swap is settled either by physical delivery or in cash. In the case of physical delivery, the buyer has the right to deliver the bond to the seller in exchange of its par value. If the terms of the contract requires a cash settlement, then company 'B' would pay an amount equal to the difference between the face value and the market value of the bond issued by the reference company. This amount of cash corresponds to the Loss Given Default and has to be evaluated at some specified number of days after default. In exchange of this insurance offered by company 'B', company 'A' periodically pays to company 'B' a fixed amount S, called the CDS spread or premium. A default also requires an accrued credit-swap premium by the buyer if the termination of the contract is triggered by a credit event.

Let us now move to the pricing of CDS contracts. As is the case for risky bonds, we assume that the face value of the reference obligation is equal to \$1. We still assume that trading follows a discrete time sequence of dates denoted by $\{t, t+1, ..., t+n\}$. The premium of the CDS, S, is paid every p periods, the coupon dates follow then the sequence $\{t, t+p, t+2p, ..., t+n\}$. The contract starts at time t and has a maturity equal to n periods. If default occurs between two consecutive trading dates, then the recovery payment is received at the next trading date following the default time. As is usually the case, we use the assumption of RFV to describe the recovery payment upon default.

Let us consider two consecutive coupon dates $]t+(k-1)p, t+kp]$. Assuming no default prior to $t+(k-1)p$, the buyer will pay the premium at $t+kp$ if no credit event occurs in the interval $]t+(k-1)p, t+kp]$. If between any two consecutive trading dates $t+i-1$ and $t+i$, where $(k-1)p+1 \leq i \leq kp$, a credit event is documented, then the buyer receives an amount of cash L_{t+i} and pays the accrued credit-swap premium $\frac{i}{p}S$. The payoff in the interval $]t+(k-1)p, t+kp]$ of the buyer can be summarized in this diagram

No default between $t+(k-1)p$ and $t+kp$	$-S$
Default occurs between two consecutive trading dates in $[t+(k-1)p, t+kp]$	$L_{t+i} - \frac{i}{p}S$

The discounted payoff of the buyer between $t+(k-1)p$ and $t+kp$ is then

$$-M_{t,t+kp}\mathbb{I}_{(\tau>t+kp)} \times S + \sum_{i=(k-1)p+1}^{kp} M_{t,t+i}\mathbb{I}_{(t+i-1<\tau<t+i)}\left(L_{t+i} - \frac{i}{kp}S\right) \qquad (3.14)$$

Applying this reasoning between t and $t+n$, the price of the CDS contract is established in the following proposition.

Proposition 2 *The discounted value, at time t, of a CDS contract with a maturity equal to $t+n$ is*

$$\begin{aligned} CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} \pi\left(t,t+kp\right) \\ &= \sum_{k=1}^{\frac{n}{p}} -E_t\left[M_{t,t+kp}\exp\left(-\sum_{i=1}^{kp}\Lambda_{t+i}\right)\right]\times S \\ &\quad +\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[\xi_i\left(L_{t+i}-\frac{i}{kp}S\right)\right], \end{aligned} \tag{3.15}$$

where $\pi\left(t,t+kp\right)$ is the expected discounted payoff at time t for the buyer between two consecutive coupon dates $t+(k-1)p$ and $t+kp$, with $1\leq k\leq\frac{n}{p}$

$$\begin{aligned} \pi\left(t,t+kp\right) &= -E\left[M_{t,t+kp}\mathbb{I}_{(\tau>t+kp)}\mid\mathcal{F}_t\right]\times S \\ &\quad +\sum_{i=(k-1)p+1}^{kp} E\left[M_{t,t+i}\mathbb{I}_{(t+i-1<\tau<t+i)}\left(L_{t+i}-\frac{i}{kp}S\right)\mid\mathcal{F}_t\right], \end{aligned} \tag{3.16}$$

and

$$\xi_i = M_{t,t+i}(\exp(-\mathbb{I}_{(j>0)}\sum_{j=1}^{i-1}\Lambda_{t+j})-\exp(-\sum_{j=1}^{i}\Lambda_{t+j})). \tag{3.17}$$

The CDS spread is computed such that the discounted value of the CDS is equal to zero

$$S=\frac{\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp}E_t\left[\xi_i L_{t+i}\right]}{\sum_{k=1}^{\frac{n}{p}}E_t\left[M_{t,t+kp}\exp\left(-\sum_{i=1}^{kp}\Lambda_{t+i}\right)\right]+\sum_{i=(k-1)p+1}^{kp}E_t\left[\xi_i\frac{i}{kp}\right]}, \tag{3.18}$$

Proof. See Appendix. ■

The discrete time specification and the assumption of a recovery payment at the next trading date following default easily accommodate the existence of accrued credit-swap premium. As is the case for risky bonds in the previous section, the pricing formula under recovery risk is written in its most general form. It involves a finite summation of conditional expectations and allows the spread to be easily inverted once the dynamics of the state vector are specified.

Once again, it might be interesting to consider the alternative that is offered by the continuous time framework. The price of a CDS contract with an annualized premium S and an accrued credit-swap premium in a continuous time framework is

$$\begin{aligned} CDS(t,t+n) &= -\sum_{k=1}^{\frac{n}{p}}E_t^Q\left[\exp\left(-\int_t^{t+kp}\left(r_u+\lambda_u\right)du\right)\right]\times pS \\ &\quad +\int_t^{t+n}E_t^Q\left[\exp\left(-\int_t^{s}\left(r_u+\lambda_u\right)du\right)\lambda_s\left(L_s-\left(\tau-\left(t+\left(k^\star-1\right)p\right)pS\right)ds\right], \end{aligned}$$

where $(t + k^{\star}p)$ corresponds to the first coupon date that follows the credit event.

Because trading is in continuous-time, it is difficult to deal with the first coupon date following the credit event. For this reason, modified payoffs are often considered in continuous time models. One possibility is to consider that rather than receiving the compensation L upon default, the buyer is paid at the next coupon date that follows default. Under such a formulation, the accrued credit-swap premium is no longer necessary. Another solution is to simply eliminate the accrued credit swap premium from the payoff. Both assumptions result in an approximated pricing solution. Although, the impact of the approximations is negligible for small default probabilities and coupon periods (p), such is not the case if one consider a higher likelihood of default. It is worth observing that the assumption of a recovery payment at the next trading date following default in our framework provides the kind of flexibility that is needed to deal with the accrued credit-swap premium.

Let us consider the case of a payoff that does not contain accrued interests, the price of the CDS is then

$$\begin{aligned} CDS(t, t+n) &= -\sum_{k=1}^{\frac{n}{p}} E_t^Q \left[\exp\left(-\int_t^{t+kp} (r_u + \lambda_u)\, du \right) \right] \times pS \\ &\quad + \int_t^{t+n} \lambda_s L_s E_t^Q \left[\exp\left(-\int_t^{s} (r_u + \lambda_u)\, du \right) ds \right]. \end{aligned} \tag{3.19}$$

The pricing formula shares many of its features with the bond's pricing solution. The second term is in deed similar to the second term in equation (2.1). As a consequence, when the Loss Given Default is stochastic, computing the price of the CDS in a continuous time set up results in the same compuational complexity as for risky bonds, which makes the model quite difficult to implement in empirical applications.

In the next section, we characterize the conditional joint distribution of the state variables using the conditional Laplace transform. We then show that knowing the Laplace transform of the state vector turns out to be sufficient to price bonds and CDS's under recovery risk.

4 Closed Form Solutions for Prices of Risky Bonds

Building on Propositions 1 an 2, we derive closed form expressions for prices of risky bonds and CDS contratcs for the general class of affine processes. Under some technical conditions, a process is affine if its conditional Laplace transform is an exponential affine function of the current values of the state variables. The family of affine dynamics is particularly attractive in our setup and offers an analytical tractability that allows the derivation of closed form solutions. Prominent among affine processes are the Gaussian and the Markov Gamma process introduced by Gourriéroux and Jasiak (2003). Mixture of these processes can also be constructed. Discrete time affine processes have also been used by Gouriéroux, Monfort and Polimenis (2002) in order to characterize the risk free term structure as in Duffie and Kan (1996).

In this section, we show that a discrete time approach is not only able to retain much of the intuition underlying the continuous time valuation framework described above, but it also provides a great analytical tractability. We first derive the pricing solutions for an affine state vector and subsequently we focus on specific econometric formulations.

4.1 Pricing Solutions for an Affine State Vector

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and make the assumption that the discrete time economy described in the previous section is driven by a state vector Y_t in some state space $C \subset \mathbb{R}^N$. To build a term structure model, we need to specify the dynamics of the pricing kernel $M_{t,t+1} = M(Y_{t,}Y_{t+1})$ underlying the time t valuation payoffs at time $t+1$, the hazard rate $\Lambda_{t+1} = \Lambda(Y_{t+1})$ and the loss given default $L_{t+1} = L(Y_{t+1})$. Toward this goal, we make the following assumptions:

Assumption 1: The time t price of any security with an exponential payoff is for any $N \times 1$ real vector α

$$E_t\left[M_{t,t+1} \exp\left(\alpha' Y_{t+1}\right)\right] = \exp(a'(\alpha) Y_t + b(\alpha), \tag{4.1}$$

where $a(\alpha)$ is a $1 \times N$ real vector and $b(\alpha)$ is a scalar.

Assumption 2: Λ_t and $\mathcal{L}_t$, where $\mathcal{L}_t = -\log(L_t)$, are expressed as follows

$$\begin{bmatrix} \Lambda_t \\ \mathcal{L}_t \end{bmatrix} = \begin{bmatrix} \gamma Y_t \\ \boldsymbol{\phi} Y_t \end{bmatrix}. \tag{4.2}$$

The first assumption states that the pair $(Y_{t+1}, M_{t,t+1})$ is affine. The no arbitrage restriction on the pricing kernel implies that

$$r_t = -a\left(0_{\mathbb{R}^N}\right) Y_t - b\left(0_{\mathbb{R}^N}\right). \tag{4.3}$$

The second assumption implies that each component of the vectors Λ_t and $\mathcal{L}_t$ is a linear combination of the state variables. The elements of γ and ϕ control the correlation between the default intensity and the Loss Given Default. The source of this correlation depends on the state variables that the hazard rate and the Loss Given Default have in common. A key benefit of this assumption is that it allows for firm specific or idiosyncratic factors that drive the default risk and the recovery rate. This corresponds to the observation of Acharya, Bharath and Srinivasan (2004) that modeling the stochastic nature of the recovery rate must take into account firm-specific factors as well as industry-specific factors. The affine specification also accomodates the inclusion of observable macroeconomic variables. Recent empirical works show that macro variables play an important role in both the risk free and the credit spread term structure and it turns out that including macro variables in standard no-arbitrage models adds to the understanding of the relationship between economic business cycles and the term structures of Treasury yields and credit spreads. Ang and Piazzesi (2003) include macro variable in a discrete time affine Gaussian model and prove that better yields forecasts are obtained when macro factors are inlcluded. Amato and Luisi (2005) find a significant impact of macro variables on the term structure of credit spreads. As pointed out by Piazzesi (2003), discrete time models are able to incorporate higher order lags of macro variables, a feature that can be exploited in our framework.

We now turn to the pricing of defaultable securities in an affine set up. Using assumptions (4.1) and (4.2), we characterize the conditional probability distribution of $(\sum_{i=1}^{p} Y_{t+i}, Y_{t+p})$ and $(\sum_{i=1}^{p} Y_{t+i}, Y_{t+1})$ via their Laplace transform in the following proposition. We then show that once these Laplace transforms are known analytically, the pricing problem described in equations (3.11) and (3.12) yields an explicit solution.

Proposition 3 *Assuming that* (4.1) *and* (4.2) *hold, then for any* $p \geq 2$:

1. *The conditional Laplace transform of* $\left(M_{t,t+p}, \sum_{i=1}^{p} Y_{t+i}, Y_{t+p}\right)$ *is given by*

$$\begin{aligned} G_{t,p}^{1}(\alpha,\beta) &\equiv E_t\left[M_{t,t+p}\exp\left(\alpha'\sum_{i=1}^{p} Y_{t+i} + \beta' Y_{t+p}\right)\right] \\ &= \exp\left(A_{1,p}(\alpha,\beta) Y_t + B_{1,p}(\alpha,\beta)\right), \end{aligned} \tag{4.4}$$

where A_1 *and* B_1 *are computed recursively as follows*

$$A_{1,i}(\alpha,\beta) = a\left(A_{1,i-1} + \alpha\right) \text{ and } A_{1,1} = a(\alpha+\beta). \tag{4.5}$$

$$B_{1,i}(\alpha,\beta) = B_{1,i-1} + b(A_{1,i-1} + \alpha) \text{ and } B_{1,1} = b(\alpha+\beta). \tag{4.6}$$

2. *The conditional Laplace transform of* $\left(M_{t,t+p}, \sum_{i=1}^{p} Y_{t+i}, Y_{t+1}\right)$ *is given by*

$$\begin{aligned} G_{t,p}^{2}(\alpha,\beta) &= \equiv E_t\left[M_{t,t+p}\exp\left(\alpha\sum_{i=1}^{p} Y_{t+i} + \beta Y_{t+1}\right)\right] \\ &= \exp\left(A_{2,p}(\alpha,\beta) Y_t + B_{2,p}(\alpha,\beta)\right), \end{aligned} \tag{4.7}$$

where

$$A_{2,p}(\alpha,\beta) = a\left(\alpha+\beta+A_{2,p-1}(\alpha)\right) \text{ and } B_{2,p}(\alpha,\beta) = B_{2,p-1} + b\left(\alpha+\beta+A_{2,p-1}(\alpha)\right), \tag{4.8}$$

and for any $i = 2, ..., p-1$, A_2 *and* B_2 *are computed recursively as follows:*

$$A_{2,i}(\alpha) = a\left(A_{2,i-1} + \alpha\right) \text{ and } A_{1,1} = a(\alpha). \tag{4.9}$$

$$B_{2,i}(\alpha) = B_{2,i-1}(\alpha) + b\left(\alpha + A_{2,i-1}(\alpha)\right) \text{ and } B_{1,1} = b(\alpha). \tag{4.10}$$

Proof. See Appendix ■

With $\alpha = \beta = 0_{\mathbb{R}^N}$, the Laplace transforms in (4.4) and (4.7) give the price at t of a risk free bond with maturity $t+n$:

$$\begin{aligned} B(t,n) &= G_{t,p}^{1}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) = G_{t,p}^{2}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) \\ &= \exp\left(A_{1,n}'(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) Y_t + B_{1,n}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N})\right) \end{aligned} \tag{4.11}$$

The term structure of the risk free rate is affine, provided that the conditional Laplace transform (4.4) is known analytically. This result gives the discrete time counterpart of the family of affine diffusions processes characterized by Duffie and Kan (1996) and generalized later to the case of affine jump diffusions processes by Duffie, Pan and Singleton (2000). The Laplace transform provides the right tools to compute closed form solutions for the prices of risky bonds under the assumptions of RT and RFV and CDS contracts. These formulas are established in the following proposition.

Proposition 4 *If the state vector Y_t follows an affine process whose Laplace transform is known analytically, we obtain the following pricing solutions by combining* (4.4) *and* (4.7) *with the general solutions derived in* (3.11) *and* (4.14)

1. *Under the assumption of RT, the price at t of a risky bond with maturity $t+n$ is*

$$\begin{aligned}\overline{B}^{RT}(t,t+n) &= G^1_{t,n}(0_{\mathbb{R}^N},0_{\mathbb{R}^N}) - G^2_{t,n}(0_{\mathbb{R}^N},-\phi) + G^1_{t,n}(-\gamma,-\phi) \\ &+\sum_{p=1}^{n-1}\left[\exp\left(B_{1,n-p}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)\right)G^1_{t,p}\left(-\gamma,\left(A_{1,n-p}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-\phi\right)\right)\right. \\ &\left.-\exp\left(B_{2,n-p}\left(0_{\mathbb{R}^N},-\phi\right)\right)G^1_{t,p}(-\gamma,A_{2,n-p}\left(0_{\mathbb{R}^N},-\phi\right))\right]. \end{aligned} \tag{4.12}$$

2. *Under the assumption of RFV, the price at t of a risky bond with maturity $t+n$ is*

$$\begin{aligned}\overline{B}^{RFV}(t,t+n) &= G^1_{t,n}\left(-\gamma,0_{\mathbb{R}^N}\right) + G^1_{t,1}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right) - G^1_{t,1}\left(-\gamma,0_{\mathbb{R}^N}\right) - G^1_{t,1}\left(-\phi,0_{\mathbb{R}^N}\right) \\ &+G^1_{t,1}\left(-\left(\gamma+\phi\right),0_{\mathbb{R}^N}\right) + \sum_{p=2}^{n}\left[\exp(b\left(0_{\mathbb{R}^N}\right))G^1_{t,p-1}\left(-\gamma,a\left(0_{\mathbb{R}^N}\right)\right)\right. \\ &-\exp(b(-\gamma))G^1_{t,p-1}(-\gamma,a\left(-\gamma\right)) - \exp(b(-\phi))G^1_{t,p-1}(-\gamma,a\left(-\phi\right)) \\ &\left.+\exp(b(-\left(\gamma+\phi\right)))G^1_{t,p-1}(-\gamma,a\left(-\left(\gamma+\phi\right)\right))\right]. \end{aligned} \tag{4.13}$$

3. *The price, at time t, of a CDS contract with maturity $t+n$ is*

$$\begin{aligned} CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -G^1_{t,kp}\left(-\gamma,0_{\mathbb{R}^N}\right)\times S \\ &+\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} G^1_{t,i}\left(-\gamma,\gamma-\phi\right) - G^1_{t,i}\left(-\gamma,-\phi\right) \\ &+\sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp}\left(G^1_{t,i}\left(-\gamma,\gamma\right) - G^1_{t,i}\left(-\gamma,0_{\mathbb{R}^N}\right)\right)\frac{i}{kp}S. \end{aligned} \tag{4.14}$$

Proof. See Appendix ■

Proposition 4 shows that the closed form expressions are rather simple to compute since they involve a finite summation. Once the Laplace transforms (4.4) and (4.7) are known analytically, it is straightforward to calculate expressions (4.12) and (4.13) as opposed to the continuous time case where a time discretization is often required to approximate several integrals. From a practical perspective, the observation that solving the pricing problem (3.11) - (4.14) reduces to the computation of two Laplace transforms is potentially interesting. It only requires the computation of A_1, A_2, B_1 and B_2 and prices of defaultable can then be obtained via these terms. As is the case in continuous time, the number of recursions is linear with the number of time steps. However, the number of terms to be computed does not increase with the number of factors. A multi-factor model is thus still tractable.

Under the assumption of RMV, the price of the risky bond does not admit a closed form expression. However, it can be computed using an efficient Monte Carlo procedure once the dynamics of the state variables are specified. It must be noted, as discussed previously, that unlike the continuous time pricing solution under RMV, the discrete time setup allows us to distinguish between the recovery rate and the hazard rate.

The correlation structure between the short rate, the default risk and the recovery rate is captured via the components of the Laplace transform, $A_{i,p}(\alpha, \beta)$ and $B_{i,p}(\alpha, \beta)$. This means that the value of the risky debt depends on both the current values of the state variables, namely the short rate, the default risk and the recovery rate, and the correlations among these variables. In other words, a bondholder's recovery may depend on the risk free term structure and the economic conditions. In a recession, when default rates tend to be high and the risk free rate is low, the recovery rate tends to decrease. This could result in a positive correlation between the loss given default and the default risk and a negative correlation with the risk free rate (see Altman (2001)). The model is able to accommodate flexible structures of correlations among the state variable as long as the conditional Laplace transform (4.1) is analytically known.

More importantly, the closed form solutions derived previously hold for a large class of discrete time processes, which makes the model quite powerful and able to provide some new insights about the term structure of corporate bonds. In the next section, we discuss several econometric specifications of the pricing kernel and the state vector. In each of these examples, the calculations are simple and do not require any numerical approximations.

Example 1 ***A Gaussian VAR(1) model***

In this example, we assume that the pricing kernel $M_{t,t+1} = M(Y_t, Y_{t+1})$ is an exponential affine function of Y_t and Y_{t+1}[4]

$$M_{t,t+1} = \exp(\gamma_1 Y_{t+1} - \gamma_2 Y_t). \tag{4.15}$$

We examine the case of a Gaussian model where the state vector follows an VAR(1) process.

$$Y_{t+1} = \boldsymbol{\delta} + \boldsymbol{\theta} Y_t + \boldsymbol{\varepsilon}_{t+1} \tag{4.16}$$

where

$$\begin{gathered}\boldsymbol{\delta} = \begin{bmatrix} \delta_1 & \ldots & \delta_n \end{bmatrix}, \boldsymbol{\theta} = diag \begin{bmatrix} \theta_1 & \ldots & \theta_n \end{bmatrix}, \boldsymbol{\varepsilon}_{t+1} \mid \mathcal{G}_t \sim N(0, \Sigma), \\ \Sigma = cov\ (\boldsymbol{\varepsilon}_{t+1}) \text{ and } cov(\varepsilon_{t+i}, \varepsilon_{t+j}) = 0, \forall\ i \neq j. \end{gathered} \tag{4.17}$$

The conditional Laplace transform (4.1) is given by

$$E_t \left[M_{t,t+1} \exp(\alpha' Y_{t+1}) \right] = \exp((\boldsymbol{\theta} (\alpha + \gamma_1) - \gamma_2)' Y_t + \frac{1}{2} (\alpha + \gamma_1)' \Sigma (\alpha + \gamma_1) + (\alpha + \gamma_1)' \boldsymbol{\delta}) \tag{4.18}$$

which implies

$$a(\alpha) = (\boldsymbol{\theta} (\alpha + \gamma_1) - \gamma_2)' \text{ and } b(\alpha) = \frac{1}{2} (\alpha + \gamma_1)' \Sigma (\alpha + \gamma_1) + (\alpha + \gamma_1)' \boldsymbol{\delta}) \tag{4.19}$$

[4]This specification is obtained when the representative agent has a power or a log utility function.

The Laplace transforms (4.4) and (4.7) can easily be computed using the following recursions

$$A_{1,i}(\alpha,\beta) = (\boldsymbol{\theta}(\alpha + A_{1,i}(\alpha,\beta) + \gamma_1) - \gamma_2)' \text{ and } A_{1,1} = (\boldsymbol{\theta}(\alpha + \beta + \gamma_1) - \gamma_2)' \tag{4.20}$$

$$B_{1,i}(\alpha,\beta) = B_{1,i-1} + \frac{1}{2}(A_{1,i-1} + \alpha + \gamma_1)'\Sigma(A_{1,i-1} + \alpha + \gamma_1) + (A_{1,i-1} + \alpha + \gamma_1)'\boldsymbol{\delta} \tag{4.21}$$

$$\text{and } B_{1,1} = \frac{1}{2}(\alpha + \beta + \gamma_1)'\Sigma(\alpha + \beta + \gamma_1) + (\alpha + \beta + \gamma_1)'\boldsymbol{\delta} \tag{4.22}$$

$$A_{2,i}(\alpha,\beta) = (\boldsymbol{\theta}(\alpha + A_{2,i-1} + \gamma_1) - \gamma_2)' \text{ and } A_{2,1} = (\boldsymbol{\theta}(\alpha + \gamma_1) - \gamma_2)' \tag{4.23}$$

$$B_{2,i}(\alpha,\beta) = B_{2,i-1} + \frac{1}{2}(A_{2,i-1} + \alpha + \gamma_1)'\Sigma(A_{1,i-1} + \alpha + \gamma_1) + (A_{2,i-1} + \alpha + \gamma_1)'\boldsymbol{\delta} \tag{4.24}$$

$$\text{and } B_{2,1} = \frac{1}{2}(\alpha + \gamma_1)'\Sigma(\alpha + \gamma_1) + (\alpha + \gamma_1)'\boldsymbol{\delta}), \tag{4.25}$$

$A_{2,n}(\alpha,\beta)$ and $B_{2,n}(\alpha,\beta)$ are computed as follows

$$A_{2,n}(\alpha,\beta) = (\boldsymbol{\theta}(\alpha + A_{2,i-1} + \beta + \gamma_1) - \gamma_2)' \text{ and}$$

$$B_{2,n}(\alpha,\beta) = B_{2,n-1} + \frac{1}{2}(A_{2,i-1} + \alpha + \beta + \gamma_1)'\Sigma(A_{1,i-1} + \alpha + \beta + \gamma_1) + (A_{2,i-1} + \alpha + \beta + \gamma_1)'\boldsymbol{\delta}$$

The Gaussian AR(1) specification was previously used by Turnbull and Milne (1991) in order to model the risk free term structure. The Gaussian AR(1) results in a mean reverting process for the risk free the short rate and can be seen as the discrete time equivalent of the Vasicek (1977) model. Under this specification, the price of a risk free bond with maturity $t+n$ is

$$B(t,n) = \exp\left(A_{1,n}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N})\, Y_t + B_{1,n}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N})\right)$$

Using the recursions (4.20)-(4.24) with the pricing solutions in (4.12) and (4.13), we get the analytical expressions for prices of risky bonds under the RT and RFV assumptions. Although very tractable, this model is known to perform relatively poorly when it comes to estimation or calibration of Treasury or corporate bond yield spreads in the reduced-form framework. An extended version, called the essentially affine Gaussian model, usually performs better. We discuss this model in the next example.

Example 2 ***A Gaussian Essentially Affine Model***

In this example, we consider the discrete time essentially affine Gaussian model. This model was originally proposed by Duffee (2002) as an extension of the standard Gaussian model. It allows for a market price of risk that is affine in the state variables, a feature that gives the model the ability to produce good forecasts of the risk free yields. When used in discrete time, this model easily accomodates the inclusion of macroeconomic variables (see Ang and Piazessi (2003) for Treasury yields and Amato and Luisi (2005) for the term structure of credit spreads), which significantly improves its empirical performance.

We assume that the pricing kernel is parametrized as in Ang and Piazzesi (2003)

$$
\begin{aligned}
M_{t,t+1} &= \exp(-\varphi_0 - \varphi_1' Y_t - \frac{1}{2} p_t' \Sigma p_t - p_t' \boldsymbol{\varepsilon}_{t+1}) \text{ and} && (4.26) \\
p_t &= p_0 + p_1 Y_t, && (4.27)
\end{aligned}
$$

where p_0 is $N \times 1$ vector and p_1 is a $N \times N$ matrix.

The state vector is assumed to follow a Gaussian VAR(1) as in equation (4.16). The time varying vector λ_t represents the so called market price of risk and the short rate implied by this specification is an affine function of the state vector

$$
r_t = \varphi_0 + \varphi_1' Y_t
$$

The main advantage of this specification is that it is able to generate time varying term premia which is an important stylized fact of the risk free term structure. However this model differentiates itself from the standard Gaussian formulation in that the pricing kernel is not an affine function of (Y_t, Y_{t+1}) and the time varying market price of risk shows up explicitly. Despite these differences, the fact that condition (4.1) holds allows us to compute the Laplace transforms $G^1_{t,n}$ and $G^2_{t,n}$ and price risky bonds. The conditional Laplace transform (4.1) is

$$
\begin{aligned}
E_t\left[M_{t,t+1} \exp(\alpha' Y_{t+1})\right] &= \exp\left(\left(-\varphi_0 + \alpha' \left(\boldsymbol{\delta} + \Sigma p_0\right) + \frac{1}{2}\alpha' \Sigma \alpha\right)\right. \\
&\quad \left. + \left(\alpha' \boldsymbol{\theta} - \varphi_1' + \alpha' \Sigma p_1\right) Y_t\right).
\end{aligned}
$$

This implies that $a(\alpha)$ and $b(\alpha)$ are written as follows

$$
a(\alpha) = \left(\alpha' \boldsymbol{\theta} - \varphi_1' + \alpha' \Sigma \lambda_1\right) \text{ and } b(\alpha) = -\varphi_0 + \alpha' \left(\boldsymbol{\delta} + \Sigma \lambda_0\right) + \frac{1}{2}\alpha' \Sigma \alpha
$$

Using the recursions (4.4) and (4.7) with the pricing solutions in (4.12) and (4.14), we get the analytical expressions for prices of risky bonds and CDS contracts under recovery risk. The correlation structure in this model clearly shows up in the terms $B_{1,i}(\alpha, \beta)$ and $B_{2,i}(\alpha, \beta)$ via the covariance matrix of the state vector Σ. The Gaussian specification allows the correlation between the state variables to be directly modeled. As is the case for the univariate model, the Gaussian AR(1) can also be seen as the discrete time equivalent of the multivariate Ornstein-Uhlenbeck process. The model easily accommodates a multifactor set up which is not the case for continuous time models. These models admit closed form expressions but at the cost of increased complexity which makes any empirical implementation very difficult (see Bakshi, Madan and Zhang (2002)). This additional computational cost is avoided here.

Example 3 ***A Markov Gamma model***

We still maintain the assumption that the pricing kernel $M_{t,t+1} = M(Y_t, Y_{t+1})$ is an exponential affine function of Y_t and Y_{t+1}

$$
M_{t,t+1} = \exp(\gamma_1' Y_{t+1} - \gamma_2' Y_t).
$$

We now consider a Markov Gamma process for the state variables. The Markov Gamma process can be thought as the analogue in discrete time of the continuous time square root process (see

Gouriéroux and Jasiak (2002)). This specification was used by Nieto-Barajas and Walker (2002) in order to model hazard rates in discrete time. Formally, a univariate process $\{u_{t+i}, i = 0, ..., n\}$ follows a Markov Gamma process, if

$$\frac{u_{t+1}}{\eta} \mid \mathcal{G}_t \sim \gamma(\delta + Z_t, 1) \text{ and } Z_t \sim \mathcal{P}(\theta \frac{u_t}{\eta})$$

where $\gamma(\delta + Z_t, 1)$ denotes the Gamma distribution with parameters $\delta + Z_t$ and 1, $\mathcal{P}(\theta \frac{\varepsilon_t}{\eta})$ is the Poisson distribution with intensity equal to θ and Z_{t+i} is independent of $Z_{t+j}, \forall\ i \neq j$.

The Markov Gamma process does not admit an AR(1) representation as is the case for the Gaussian process. However, conditional moments of this process can be derived using the conditional Laplace transform. The latter can be written as follows for any scalar α

$$E_t\left[\exp(\alpha u_{t+1})\right] = E_t\left[E\left[\exp(\alpha u_{t+1}) \mid Z_t\right]\right].$$

Using the Laplace transform of a Gamma distributed random variable (see Johnson et al (1995)), we get

$$E_t\left[\exp(\alpha u_{t+1})\right] = \exp\left(-\delta \log\left(1 - \eta\alpha\right) + \frac{\theta\alpha}{1 - \eta\alpha} u_t\right).$$

Now, following Gouriéroux and Jasiak (2002), we build a multivariate Markov process by assuming that each component of the state vector Y_t follows a Markov Gamma process and that they are mutually independent

$$\frac{Y_{k,t+1}}{\eta_k} \mid \mathcal{G}_t \sim \gamma(\delta_k + Z_{k,t}, 1) \text{ and } Z_{k,t} \sim \mathcal{P}(\theta_k \frac{Y_{k,t}}{\eta_k}), \text{ for any } k = \{1, .., n\}.$$

It is important to notice that the assumption of independence between the state variables constrains the correlation to be captured via $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, as opposed to the Gaussian case where the correlation among the state variable is explicitly modeled. However, the conditional variance in the Gamma Markov model depends on each $Y_{k,t}$. As it is the case in continuous time affine models, the choice between the Gaussian specification and the Gamma Markov model involves a trade-off between a conditional variance driven by the state variables and a total flexibility regarding the correlations among the state variables[5].

The conditional Laplace transform (4.1) is

$$E_t\left[M_{t,t+1} \exp(\alpha' Y_{t+1})\right] = \exp\left(\sum_{k=1}^{n} a_k\left(\alpha_k\right) Y_{k,t} + b_k\left(\alpha_k\right)\right), \tag{4.28}$$

where $a_k(\alpha)$ and $b_k(\alpha)$ are computed follows:

$$a_k\left(\alpha\right) = \frac{\theta_k\left(\alpha_k + \gamma_{1k}\right)}{1 - \eta_k\left(\alpha_k + \gamma_{1k}\right)} - \gamma_{2k} \text{ and } b_k\left(\alpha\right) = -\delta_k \log\left(1 - \eta_k\left(\alpha_k + \gamma_{1k}\right)\right), \text{ for any } k = \{1, ..., n\}. \tag{4.29}$$

Assembling these ingredients, the Laplace transforms (4.4) and (4.7) can be computed as follows

[5] See Dai and Singleton (2000) for a detailed discussion.

$$
\begin{aligned}
G_{t,p}^{1}(\alpha,\beta) &= \exp\left(\sum_{k=1}^{n} A_{1,p}^{k}(\alpha,\beta)\,Y_{k,t} + B_{1,p}^{k}(\alpha,\beta)\right) \text{ and} \\
G_{t,p}^{2}(\alpha,\beta) &= \exp\left(\sum_{k=1}^{n} A_{2,p}^{k}(\alpha,\beta)\,Y_{k,t} + B_{2,p}^{k}(\alpha,\beta)\right),
\end{aligned}
$$

using the following recursions for any $k = \{1, ..., n\}$

$$
A_{1,i}^{k}(\alpha,\beta) = \left(\frac{\theta_k\left(\alpha_k + \gamma_{1k} + A_{1,i-1}^{k}(\alpha,\beta)\right)}{1 - \eta_k\left(\alpha_k + \gamma_{1k} + A_{1,i-1}^{k}(\alpha,\beta)\right)}\right) - \gamma_{2k} \text{ and } A_{1,1}^{k} = \frac{\theta_k\left(\alpha_k + \gamma_{1k} + \beta_k\right)}{1 - \eta\left(\alpha_k + \gamma_{1k} + \beta_k\right)} - \gamma_{2k}
$$

$$
\begin{aligned}
B_{1,i}^{k}(\alpha,\beta) &= B_{1,i-1}^{k} - \delta_k \log\left(1 - \eta_k\left(\alpha_k + \gamma_{1k} + A_{1,i-1}^{k}(\alpha,\beta)\right)\right) \text{ and} \\
B_{1,1}^{k} &= -\delta_k \log\left(1 - \eta_k\left(\alpha_k + \gamma_{1k} + \beta_k\right)\right)
\end{aligned}
$$

$$
A_{2,i}^{k}(\alpha) = \frac{\theta_k\left(\alpha_k + \gamma_{1k} + A_{1,i-1}^{k}(\alpha,\beta)\right)}{1 - \eta_k\left(\alpha_k + \gamma_{1k} + A_{1,i-1}^{k}(\alpha,\beta)\right)} - \gamma_{2k} \text{ and } A_{2,1}^{k} = \frac{\theta_k\left(\alpha_k + \gamma_{1k}\right)}{1 - \eta_k\left(\alpha_k + \gamma_{1k}\right)} - \gamma_{2k}
$$

$$
\begin{aligned}
B_{2,i}^{k}(\alpha) &= B_{1,i-1}^{k} - \delta_k \log\left(1 - \eta_k\left(\alpha_k + \gamma_{1k} + A_{1,i-1}^{k}(\alpha,\beta)\right)\right) \text{ and} \\
B_{2,1}^{k} &= -\delta_k \log\left(1 - \eta_k\left(\alpha_k + \gamma_{1k}\right)\right),
\end{aligned}
$$

$A_{2,n}(\alpha,\beta)$ and $B_{2,n}(\alpha,\beta)$ are computed as follows

$$
A_{2,n}(\alpha,\beta) = \frac{\theta_k\left(\alpha_k + \gamma_{1k} + \beta_k + A_{1,i-1}^{k}(\alpha)\right)}{1 - \eta_k\left(\alpha_k + \gamma_{1k} + \beta_k + A_{1,i-1}^{k}(\alpha)\right)} \text{ and}
$$

$$
B_{2,n}(\alpha,\beta) = B_{1,i-1}^{k} - \delta_k \log\left(1 - \eta_k\left(\alpha_k + \gamma_{1k} + \beta_k + A_{1,i-1}^{k}(\alpha)\right)\right).
$$

In particular, the risk free short rate is implied by the following equality

$$
\exp(-r_t) = E_t\left[M_{t,t+1}\right],
$$

The dynamic of the risk free short rate is then given by

$$
r_t = \sum_{k=1}^{n} \delta_k \log\left(1 + \eta\boldsymbol{\gamma}_{1k}\right) - \left(\frac{\eta_k \gamma_{1k}}{\eta_k \boldsymbol{\gamma}_{1k} + 1} - \boldsymbol{\gamma}_{2k}\right) Y_{k,t}. \tag{4.30}
$$

Combining these Laplace transforms with the pricing solutions in (4.12) and (4.13), we get the analytical expressions for prices of risky bonds under the RT and RFV assumptions for the multifactor factor Markov Gamma model.

4.2 A Numerical Example

As an illustration of these pricing results, we investigate the computation of bond prices and credit spreads for the four-factor Gaussian AR(1) model (4.16) under the RT and RFV assumptions. Given the analytical expressions for the price of the risky bond, one can compute the yield spread. This latter is the difference between the yield on a risky bond and the yield on a Treasury bond with the same maturity. In addition to illustrate the ease of computing these prices, we explore the sensitivity of the bond price to the current value of the recovery rate. More precisely, we assume the following parameterization

$$\begin{bmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \boldsymbol{\theta}_3 \\ \boldsymbol{\theta}_4 \end{bmatrix} = \begin{bmatrix} 0.050 \\ 0.015 \\ 0.095 \\ 0.448 \end{bmatrix}, \begin{bmatrix} \delta_1 \\ \delta_3 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 0.050 \\ 0.050 \\ 0.17 \\ 0.67 \end{bmatrix}, \tag{4.31}$$

$$\Sigma = \begin{bmatrix} 0.2 & 0.15 & -0.15 & -0.1 \\ 0.15 & 0.1 & 0.07 & -0.015 \\ -0.15 & 0.07 & 0.05 & 0.15 \\ -0.1 & -0.015 & 0.15 & 0.15 \end{bmatrix},$$

$$\gamma_1 = \gamma_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \gamma \\ \phi \end{bmatrix} = -\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure 1 shows the term structure of credit spreads for varying values of the state variable that controls the level of the recovery rate, Y_4. The initial values of the three first components of the state vector are $\begin{pmatrix} Y_1 = 0.15 & Y_2 = 0.051 & Y_3 = 0.09 \end{pmatrix}'$. The initial value of the fourth state variable Y_4 is set equal to $0.1, 0.5$ and 0.9. Both assumptions result in a term structure that is decreasing in the current value of the recovery rate. As the value of the fourth state variable increases from 0.1 to 0.9, the current value of the loss given default decreases. As a result, the credit spreads significantly decrease. In general, the magnitude of the decrease depends on the bond rating. It is worth observing that the credit spreads generated by this particular choice of parameters are rather high, which indicates a high yield bond. For such a bond, the impact of the initial level of the recovery rate is significant. As the level of the fourth state variable increases from 0.1 to 0.9, the current value of the loss given default decreases and the maximum level of the credit spreads drops from approximately 13% to 11% for the assumption of RFV and from 10% to 8% for the assumption of RT. This corresponds to the observation of Altman (2001) that recovery rates have a significant impact on high-yield bonds. Another interesting observation is that the model generates a humped shaped yield spread curve under the RFV assumption and a monotone decreasing curve under the RT assumption. Although the shape of the term structure depends on the particular parameterization of the model, it is important to notice that the model is able to generate both patterns.

5 Conclusion and Directions for Future Research

Recent empirical evidence illustrates the need for modeling risk associated with the recovery rate in addition to the probability of default. However, while significant progress has been made in modeling the term structure of defaultable bonds, relatively few studies have modeled the time varying nature of the recovery rate. This paper proposes a general equilibrium methodology for modeling the term structure of defaultable bonds that jointly models the recovery rate and the default probability. While most of the credit risk literature focuses on continuous time processes for modeling the term structure of defaultable bonds, this paper assumes a discrete-time economy. Interestingly, while part of the motivation for continuous time models is usually their analytical tractability, this paper demonstrates that a discrete time setup is more tractable than its continuous time counterpart when the recovery rate is allowed to vary stochastically.

We provide general pricing solutions for CDS contracts and risky bonds under three standard assumptions: RT, RFV and RMV. We then focus on the case of an economy with affine state variables, and derive closed form expressions for prices of risky bonds and CDS contracts using the conditional Laplace transform of the state variables. Availability of the conditional Laplace transform of the state vector in closed form is a sufficient condition for pricing risky bonds and CDS contracts analytically, as opposed to the continuous time case where a time discretization is often required to approximate one or more integrals even when the Laplace transform is known analytically. The family of affine discrete time process for which closed form solutions under RT and RFV assumption are available allows for potentially flexible correlation structures. Under the RMV assumption of Duffie and Singleton (1999), a closed form solution is not available, but the price of the risky bond can be computed via Monte Carlo simulation. Unlike the continuous-time case in Duffie and Singleton (1999), the model allows for explicitly modeling and identifying the time-varying nature of the recovery rate under the RMV assumption.

The model allows to empirically investigate the link between the risk free term structure, default risk and the recovery rate captured in this model. Although the Basle Committee has identified recovery risk as an important source of risk in addition to default, the impact of recovery rates on bond prices is not yet fully understood in the empirical literature. Further more, explicitly modeling recovery rates enables the model to infer the recovery rates that are implicit in bond prices and investigate which recovery assumption is best supported by market prices.

A Appendix

Proof of proposition 1

We first start by recalling how to construct a Cox process. The filtration $\{\mathcal{F}_{t+i}; i = 0, ..., n\}$ reflects the evolution of the set of state variables up to time $t + i$ and the occurrence of default. If we denote by $\{\mathcal{G}_{t+i}, i = 0, ..., n\}$ the basic filtration without the information on the occurrence of default, then $\mathcal{F}_{t+i}$ can be written as

$$\mathcal{F}_{t+i} = \mathcal{G}_{t+i} \vee \mathcal{H}_{t+i},$$

where $\mathcal{H}_{t+i}$ is the sigma field $\sigma(\tau < s, s \leq t + i)$ that holds the information on whether default occurred before $t + i$.

Formally a Cox process, also known as a doubly stochastic process driven by the tribe $\mathcal{G}_{t+i}$, is a counting process that has a Poisson distribution with intensity Λ_{t+1}, conditional on $\mathcal{G}_{t+1} \vee \mathcal{H}_t$, $\forall$ $i \geq 1$, where

$$\Lambda_{t+i} = \int_{t+i-1}^{t+i} \lambda(u)\, du < \infty$$

The default time τ is said to be doubly stochastic driven by $\{\mathcal{F}_{t+i}; i = 0, ..., n\}$ if the underlying counting process whose first jump time is τ is doubly stochastic.

For the simple case of zero recovery in the event of default, one can use the double stochasticity of the default time τ to show that

$$E\left[\mathbb{I}_{(\tau > t+i)} \mid \mathcal{G}_{t+k} \vee \mathcal{H}_t\right] = \mathbb{I}_{(\tau > t)} \exp\left(-\sum_{i=1}^{k} \Lambda_{t+i}\right).$$

The price of the risky bond under the zero-recovery assumption is then

$$\begin{aligned}
\overline{B}(t, t+n) &= E\left[M_{t,t+n}\mathbb{I}_{(\tau > t+n)} \mid \mathcal{F}_t\right] \\
&= E\left[M_{t,t+n}E\left[\mathbb{I}_{(\tau > t+n)} \mid \mathcal{G}_{t+n} \vee \mathcal{H}_t\right] \mid \mathcal{F}_t\right] \\
&= \mathbb{I}_{(\tau > t)}E\left[M_{t,t+n}\exp\left(-\sum_{i=1}^{n} \Lambda_{t+i}\right) \mid \mathcal{F}_t\right]
\end{aligned}$$

Now, recall that $\{M_{t,t+i}, i = 0, ..., n\}$ and $\{\Lambda_{t+i}, i = 0, 1, ..., n\}$ depend only on the set of state variables. Since the state variables are $\mathcal{G}_t$-measurable processes, and thus independent of $\mathcal{H}_t$, one can replace the conditioning on $\mathcal{F}_t$ by a conditioning on $\mathcal{G}_t$,which implies that

$$\begin{aligned}
\overline{B}(t, t+n) &= \mathbb{I}_{(\tau > t)}E\left[M_{t,t+n}\exp\left(-\sum_{i=1}^{n} \Lambda_{t+i}\right) \mid \mathcal{G}_t\right] \\
&= \mathbb{I}_{(\tau > t)}E_t\left[M_{t,t+n}\exp\left(-\sum_{i=1}^{n} \Lambda_{t+i}\right)\right]
\end{aligned}$$

The argument of the double stochasticity of τ is also used to derive the pricing solutions under the RT, RFV and RMV assumptions:

1. Under the RT assumption, the value of the risky bond verifies the following recursive equation

$$\overline{B}^{RT}(t,t+n) = E_t\left[M_{t,t+1}\left(\overline{B}^{RT}(t+1,t+n)\,\mathbb{I}_{(\tau>t+1)} + (1-L_{t+1})\,B(t+1,t+n)\mathbb{I}_{(\tau<t+1)}\right)\right]$$

and

$$\overline{B}^{RT}(t+n-1,t+n) = E_t\left[M_{t+n-1,t+n}\left(1-L_{t+n}\left(\mathbb{I}_{(\tau<t+n)}\right)\right)\right].$$

The same argument as in proposition 1 yields

$$\begin{aligned}\overline{B}^{RT}(t,t+n) &= E_t\left[M_{t,t+n}(1-L_{t+1})\left(1-\exp(-\Lambda_{t+1})\right)\right] \\ &\quad \underbrace{+E_t\left[M_{t,t+1}\exp\left(-\Lambda_{t+1}\right)\overline{B}^{RT}(t+1,t+n)\right]}_{K_1}.\end{aligned} \tag{A.1}$$

Using the law of iterated expectations, K_1 can be written as follows

$$\begin{aligned}K_1 &= E_t\left[M_{t,t+2}\exp(-\sum_{i=1}^{2}\Lambda_{t+i})\overline{B}^{RT}(t+2,t+n)\right] \\ &\quad +E_t\left[M_{t,t+n}\left(1-\exp(-\Lambda_{t+2})\right)\exp(-\Lambda_{t+1})(1-L_{t+2})\right].\end{aligned}$$

Equation $(A.1)$ implies that

$$\overline{B}^{RT}(t,n) = \underbrace{E_t\left[M_{t,t+2}\exp(-\sum_{i=1}^{2}\Lambda_{t+i})\overline{B}^{RT}(t+2,t+n)\right]}_{K_2}$$

$$\begin{aligned}&+E_t\left[M_{t,t+n}\left(\exp(-\Lambda_{t+1})\left(L_{t+1}-L_{t+2}\right)+\left(1-L_{t+1}\right)\right.\right. \\ &\left.\left.-\exp(-\sum_{i=1}^{2}\Lambda_{t+i})\left((1-L_{t+2})\right)\right)\right].\end{aligned} \tag{A.2}$$

K_2 can be written as follows

$$\begin{aligned}K_2 &= E_t\left[M_{t,t+3}\exp(-\sum_{i=1}^{3}\Lambda_{t+i})\overline{B}^{RT}(t+3,t+n)\right. \\ &\quad \left.+M_{t,t+n}\left((1-\exp(-\Lambda_{t+i})\right)\exp(-\sum_{i=1}^{2}\Lambda_{t+i})(1-L_{t+3})\right].\end{aligned} \tag{A.3}$$

Plugging $(A.3)$ into $(A.2)$ yields

$$\begin{aligned}\overline{B}^{RT}(t,t+n) &= E_t\left[M_{t,t+3}\exp(-\sum_{i=1}^{3}\Lambda_{t+i})\overline{B}^{RT}(t+2,t+n)\right]+E_t\left[M_{t,t+n}(1-L_{t+1})\right]\\ &+E_t\left[M_{t,t+n}\left(\sum_{p=1}^{2}\exp(-\sum_{i=1}^{p-1}\Lambda_{t+i})\left(L_{t+p}-L_{t+p+1}\right)\right.\right.\\ &\left.\left.-\exp(-\sum_{i=1}^{3}\Lambda_{t+i})\left((1-L_{t+3})\right)\right)\right].\end{aligned}$$

Continuing the recursion and taking into account the boundary condition gives

$$\begin{aligned}\overline{B}^{RT}(t,t+n) &= E_t\left[(1-L_{t+1})M_{t,t+n}\right]+E_t\left[M_{t,t+n}L_{t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right]\\ &+E_t\left[M_{t,t+n}\left(\sum_{p=1}^{n-1}\exp(-\sum_{i=1}^{p}\Lambda_{t+i})\left(L_{t+p}-L_{t+p+1}\right)\right)\right].\end{aligned}$$

2. Under the RFV assumption, the value of the risky bond verifies the following recursive equation

$$\overline{B}^{RFV}(t,t+n)=E_t\left[M_{t,t+1}\left(\overline{B}^{RFV}(t+1,t+n)\mathbb{I}_{(\tau>t+1)}+(1-L_{t+1})\,\mathbb{I}_{(\tau<t+1)}\right)\right]$$

and

$$\overline{B}^{RFV}(t+n-1,t+n)=E_{t+n-1}\left[M_{t+n-1,t+n}\left(1-L_{t+n}\left(\mathbb{I}_{(\tau<t+n)}\right)\right)\right].$$

Using the same argument as in proposition 1, we get

$$\begin{aligned}\overline{B}^{RFV}(t,t+n) &= \underbrace{E_t\left[M_{t,t+1}\exp\left(-\Lambda_{t+1}\right)\overline{B}^{RFV}(t+1,n)\right]}_{K_1}\\ &+E_t\left[M_{t,t+1}\left(1-\exp(-\Lambda_{t+1})\right)\left(1-L_{t+1}\right)\right].\end{aligned}\tag{A.4}$$

K_1 can be written as follows

$$\begin{aligned}K_1 &= E_t\left[M_{t,t+2}\exp(-\sum_{i=1}^{2}\Lambda_{t+i})\overline{B}^{RFV}(t+2,t+n)\right]\\ &+E_t\left[M_{t,t+2}\exp(-\Lambda_{t+1})\left(1-\exp(-\Lambda_{t+2})\left(1-L_{t+2}\right)\right)\right].\end{aligned}$$

Equation $(A.4)$ implies that

$$\begin{aligned}\overline{B}^{RFV}(t,t+n) &= E_t\left[M_{t,t+2}\exp(-\sum_{i=1}^{2}(\Lambda_{t+i}))\overline{B}^{RFV}(t+2,t+n)\right]\\ &+E_t\left[M_{t,t+1}\left(1-\exp(-\Lambda_{t+1})\right)\left(1-L_{t+1}\right)\right]\\ &+E_t\left[M_{t,t+2}\exp(-\Lambda_{t+1})\left(1-\exp(-\Lambda_{t+2})\right)\left(1-L_{t+2}\right)\right].\end{aligned}$$

Continuing the recursion and taking into account the boundary condition gives

$$\begin{aligned}\overline{B}^{RFV}(t,t+n) &= E_t\left[M_{t,t+n}\exp(-\sum_{i=1}^{n}\Lambda_{t+i})\right] + E_t\left[M_{t,t+1}\left(1-\exp(-\Lambda_{t+1})\right)(1-L_{t+1})\right] \\ &+ \sum_{p=2}^{n} E_t\left[M_{t,t+p}\exp\left(-\sum_{i=1}^{p-1}\Lambda_{t+i}\right)\left(1-\exp(-\Lambda_{t+p})\right)(1-L_{t+p})\right].\end{aligned}$$

3. Under the RMV assumption, using the same argument as previously, we get

$$\overline{B}^{RMV}(t,t+n) = E_t\left[\prod_{i=1}^{n} M_{t+i-1,t+i}\left[1-L_{t+i}\left(1-\exp(-\Lambda_{t+i})\right)\right]\right]. \tag{A.5}$$

Proof of proposition 2

Recall that the discount payoff, at time t, of a CDS between two consecutive coupon-date $t+(k-1)p+1$ and $t+kp$ is

$$-M_{t,t+kp}\mathbb{I}_{(\tau>t+kp)} \times S + \sum_{i=(k-1)p+1}^{kp} M_{t,t+i}\mathbb{I}_{(t+i-1<\tau<t+i)}\left(L_{t+i}-\frac{i}{kp}S\right).$$

The price of the CDS is then

$$\begin{aligned}CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -E\left[M_{t,t+kp}\mathbb{I}_{(\tau>t+kp)} \mid \mathcal{F}_t\right] \times S \\ &+ \sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[M_{t,t+i}\mathbb{I}_{(t+i-1<\tau<t+i)}\left(L_{t+i}-\frac{i}{kp}S\right) \mid \mathcal{F}_t\right],\end{aligned} \tag{A.6}$$

Using the double stochasticity of the default time, we get

$$\begin{aligned}CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -E_t\left[M_{t,t+kp}\exp\left(-\sum_{j=1}^{kp}\Lambda_{t+i}\right)\right] \times S \\ &+ \sum_{k=1}^{\frac{n}{p}}\sum_{i=(k-1)p+1}^{kp} E_t\left[\xi_i\left(L_{t+i}-\frac{i}{kp}S\right)\right],\end{aligned} \tag{A.7}$$

where

$$\xi_i = M_{t,t+i}(\exp(-\mathbb{I}_{(j>0)}\sum_{j=1}^{i-1}\Lambda_{t+j}) - \exp(-\sum_{j=1}^{i}\Lambda_{t+j})).$$

The fair spread can be inverted by setting the price equal to 0.

Proof of proposition 3

The proof of this proposition proceeds by induction, we first verify the formula for $t+1$, assume that it holds for $t+n-1$ and prove that it is also true for $t+n$.

$$\begin{aligned} G^1_{t,1}(\alpha,\beta) &\equiv E_t\left[M_{t,t+1}\exp\left((\alpha+\beta)' Y_{t+1}\right)\right] \\ &= \exp(a'(\alpha+\beta)Y_t + b(\alpha+\beta)). \end{aligned}$$

Now for $t+1$, we have

$$\begin{aligned} G^1_{t,p}(\alpha,\beta) &\equiv E_t\left[M_{t,t+1}\exp(\alpha' Y_{t+1}) E_{t+1}\left[M_{t+1,t+p}\exp(\alpha'\sum_{i=1}^{p-1} Y_{t+1+i} + \beta' Y_{t+p})\right]\right] \\ &= E_t\left[M_{t,t+1}\exp\left((A_{1,p-1}+\alpha)' Y_{t+1} + B_{1,p-1}\right)\right] \\ &= \exp\left(a'(\alpha + A_{1,p-1}) Y_t + B_{1,p-1} + b(\alpha + A_{1,p-1})\right). \end{aligned}$$

Using the same reasoning, one can derive $G^2_{t,p}(\alpha,\beta)$.

Proof of proposition 4

1. Under the assumption of RT, the price of a risky bond with maturity $t+n$ is

$$\begin{aligned} \overline{B}^{RT}(t,t+n) &= E_t\left[M_{t,t+n}(1-L_{t+1})\right] + E_t\left[M_{t,t+n}L_{t+n}\exp\left(-\sum_{j=i+1}^{n}\Lambda_{t+i}\right)\right] \\ &\quad + \sum_{p=i+1}^{n-1} E_t\left[M_{t+p,t+n}\left(M_{t,t+p}\exp\left(-\sum_{i=1}^{p}\Lambda_{t+i}\right)(L_{t+p}-L_{t+p+1})\right)\right]. \end{aligned}$$

The first three terms can be computed using the Laplace transforms in (4.4) and (4.7) :

$$E_t\left[M_{t,t+n}(1-L_{t+1})\right] = G^1_{t,n}(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}) - G^2_{t,n}(0_{\mathbb{R}^N}, -\phi)$$

and

$$E_t\left[M_{t,t+n}L_{t+n}\exp\left(-\sum_{j=i+1}^{n}\Lambda_{t+i}\right)\right] = G^1_{t,n}(-\gamma,-\phi).$$

The last term can be written as follows

$$\begin{aligned} &\sum_{p=i+1}^{n-1} E_t\left[M_{t+p,t+n}\left(M_{t,t+p}\exp\left(-\sum_{i=1}^{p}\Lambda_{t+i}\right)(L_{t+p}-L_{t+p+1})\right)\right] \\ &= \sum_{p=i+1}^{n-1} J_p - K_p. \end{aligned}$$

J_p can be decomposed as follows

$$\begin{aligned} J_p &= E_t\left[M_{t,t+p}\exp\left(-\sum_{i=1}^{p}\Lambda_{t+i}-\mathcal{L}_{t+p}\right)E_{t+p}\left[M_{t+p,t+n}\right]\right] \\ &= \exp\left(B_{1,n-p}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)\right)E_t\left[\exp\left(-\gamma\sum_{i=1}^{p}Y_{t+i}+\left(A_{1,n-p}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-\phi\right)'Y_{t+p}\right)\right] \\ &= \exp\left(B_{1,n-p}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)\right)G^1_{t,p}\left(-\gamma,\left(A_{1,n-p}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^n}\right)-\phi\right)\right). \end{aligned}$$

K_p can be written as

$$\begin{aligned} K_p &= E_t\left[M_{t,t+p}\exp\left(-\sum_{i=1}^{p}\Lambda_{t+i}\right)E_{t+p}\left[M_{t+p,t+n}\exp\left(-\mathcal{L}_{t+p+1}\right)\right]\right] \\ &= \exp\left(B_{2,n-p}\left(0_{\mathbb{R}^N},-\phi\right)\right)E_t\left[M_{t,t+p}\exp\left(-\gamma\sum_{i=1}^{p}Y_{t+i}+A'_{2,n-p}\left(0_{\mathbb{R}^N},-\phi\right)Y_{t+p}\right)\right] \\ &= \exp\left(B_{2,n-p}\left(0_{\mathbb{R}^N},-\phi\right)\right)G^1_{t,p}(-\gamma,A_{2,n-p}\left(0_{\mathbb{R}^N},-\phi\right)). \end{aligned}$$

This implies that

$$\begin{aligned} \overline{B}^{RT}(t,t+n) &= G^1_{t,n}(0_{\mathbb{R}^N},0_{\mathbb{R}^N})-G^2_{t,n}(0_{\mathbb{R}^N},-\phi)+G^1_{t,n}(-\gamma,-\phi) \\ &\quad+\sum_{p=1}^{n-1}\Big[\exp\left(B_{1,n-p}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)\right)G^1_{t,p}\left(-\gamma,\left(A_{1,n-p}\left(\alpha_1,0_{\mathbb{R}^n}\right)-\phi\right)\right) \\ &\quad-\exp\left(B_{2,n-p}\left(0_{\mathbb{R}^N},-\phi\right)\right)G^1_{t,p}(-\gamma,A_{2,n-p}\left(0_{\mathbb{R}^N},-\phi\right))\Big]. \end{aligned}$$

2. Under the assumption of RFV, the price at t of a risky bond with maturity $t+n$ is

$$\begin{aligned} \overline{B}^{RFV}(t,t+n) &= E_t\left[M_{t,t+n}\exp\left(-\sum_{i=1}^{n}\Lambda_{t+i}\right)\right]+E_t\left[M_{t,t+1}\left(1-\exp(-\Lambda_{t+1})\right)\left(1-L_{t+1}\right)\right] \\ &\quad+\sum_{p=2}^{n}E_t\left[M_{t,t+p}\exp\left(-\sum_{i=1}^{p-1}\Lambda_{t+i}\right)\left(1-\exp(-\Lambda_{t+p})\right)\left(1-L_{t+p}\right)\right]. \end{aligned}$$

Using the Laplace transforms in (4.4) and (4.7) , $\overline{B}^{RFV}(t,n)$ can be rewritten as

$$\begin{aligned} \overline{B}^{RFV}(t,t+n) &= G^1_{t,n}\left(-\gamma,0_{\mathbb{R}^N}\right)+G^1_{t,1}\left(0_{\mathbb{R}^N},0_{\mathbb{R}^N}\right)-G^1_{t,1}\left(-\gamma,0_{\mathbb{R}^N}\right) \\ &\quad-G^1_{t,1}\left(-\phi,0_{\mathbb{R}^N}\right)+G^1_{t,1}\left(-\left(\gamma+\phi\right),0_{\mathbb{R}^N}\right) \\ &\quad+\sum_{p=2}^{n}E_t\Bigg[M_{t,t+p-1}\exp(-\sum_{i=1}^{p-1}\Lambda_{t+i}) \\ &\quad E_{t+p-1}\left[M_{t+p-1,t+p}\left(1-\exp(-\Lambda_{t+p})\right)\left(1-L_{t+p}\right)\right]\Bigg]. \end{aligned}$$

It therefore follows that

$$
\begin{aligned}
\overline{B}^{RFV}(t,t+n) &= G^1_{t,n}\left(-\gamma, 0_{\mathbb{R}^N}\right) + G^1_{t,1}\left(0_{\mathbb{R}^N}, 0_{\mathbb{R}^N}\right) - G^1_{t,1}\left(-\gamma, 0_{\mathbb{R}^N}\right) - G^1_{t,1}\left(-\phi, 0_{\mathbb{R}^N}\right) \\
&\quad + G^1_{t,1}\left(-\left(\gamma+\phi\right), 0_{\mathbb{R}^N}\right) + \sum_{p=2}^{n}\left[\exp(b\left(0_{\mathbb{R}^N}\right))G^1_{t,p-1}\left(-\gamma, a\left(0_{\mathbb{R}^N}\right)\right)\right. \\
&\quad - \exp(b(-\gamma))G^1_{t,p-1}(-\gamma, a\left(-\gamma\right)) - \exp(b(-\phi))G^1_{t,p-1}(-\gamma, a\left(-\phi\right)) \\
&\quad \left. + \exp(b(-\left(\gamma+\phi\right)))G^1_{t,p-1}(-\gamma, a\left(-\left(\gamma+\phi\right)\right))\right]. \qquad (A.8)
\end{aligned}
$$

3. The price, at time t, of a CDS contract with maturity $t+n$ is

$$
\begin{aligned}
CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -E_t\left[M_{t,t+kp}\exp\left(-\sum_{j=1}^{kp}\Lambda_{t+i}\right)\right] \times S \\
&\quad + \sum_{k=1}^{\frac{n}{p}} \sum_{i=(k-1)p+1}^{kp} E_t\left[\xi_i\left(L_{t+i} - \frac{i}{kp}S\right)\right], \qquad (A.9)
\end{aligned}
$$

where

$$
\xi_i = M_{t,t+i}(\exp(-\mathbb{I}_{(j>0)}\sum_{j=1}^{i-1}\Lambda_{t+j}) - \exp(-\sum_{j=1}^{i}\Lambda_{t+j})).
$$

Once again, using the Laplace transform in (4.4), $CDS(t,t+n)$ can be rewritten as

$$
\begin{aligned}
CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -G^1_{t,kp}\left(-\gamma, 0_{\mathbb{R}^N}\right) \times S \qquad (A.10) \\
&\quad + \sum_{k=1}^{\frac{n}{p}} \sum_{i=(k-1)p+1}^{kp} E_t\left[M_{t,t+i}(\exp(-\sum_{j=1}^{i}\Lambda_{t+j} + \Lambda_{t+i})\right. \\
&\quad \left. - \exp(-\sum_{j=1}^{i}\Lambda_{t+j}))\left(L_{t+i} - \frac{i}{kp}S\right)\right]. \qquad (A.11)
\end{aligned}
$$

It therefore follows that

$$
\begin{aligned}
CDS(t,t+n) &= \sum_{k=1}^{\frac{n}{p}} -G^1_{t,kp}\left(-\gamma, 0_{\mathbb{R}^N}\right) \times S \\
&\quad + \sum_{k=1}^{\frac{n}{p}} \sum_{i=(k-1)p+1}^{kp} G^1_{t,i}\left(-\gamma, \gamma-\phi\right) - G^1_{t,i}\left(-\gamma, -\phi\right) \\
&\quad + \sum_{k=1}^{\frac{n}{p}} \sum_{i=(k-1)p+1}^{kp} \left(G^1_{t,i}\left(-\gamma, \gamma\right) - G^1_{t,i}\left(-\gamma, 0_{\mathbb{R}^N}\right)\right)\frac{i}{kp}S. \qquad (A.12)
\end{aligned}
$$

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Figure 1. Term structure of credit spreads for a four-factor Gaussian model