Discussion of: "Nonparametric Option-Implied Volatility" by Viktor Todorov

Dacheng Xiu

The University of Chicago, Booth School of Business

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Summary

- Goal: Infer spot volatility from short-dated options
- Key idea: Carr and Madan (2001) + Lévy - Khintchine formula

$$
\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}(\exp iu(x_{t+\tau} - x_t)) \\
=&1 - (u^2 + iu) \int_{\mathbb{R}} \exp((iu-1)k - iux_t) O_\tau(k) dk, \\
&\mathbb{E}_t^{\mathbb{Q}}(\exp iu(x_{t+\tau} - x_t)) \\
\approx&\exp\left\{ iu\tau a_t - \frac{u^2\tau}{2} V_t + \tau \int_{\mathbb{R}} (\exp iux - 1 - iux)\, \nu_t(dx) \right\}.
\end{aligned}
$$

- The first formula involves “estimation” and Riemann sum approximation, hence the need for certain asymptotic error bound on $|O_\tau(k_1) - O_\tau(k_2)|$ and a *good* span of strikes.
- The second formula involves asymptotic expansion of the characteristics function as $\tau \to 0$ for a *good* jump diffusion $\widetilde{x}_s$, as well as approximation of the $\mathbb{Q}$-characteristic function of x_s using that of $\widetilde{x}_s$.
- Combining the two yields the desired consistency.
- The paper also discusses adaptive choice of u and the CLT.

Overall, this is a very cool paper with many intricate calculations.

Asymptotic Expansion (AE)

- It is well known that the BS-implied volatility of an ATM option is close to the spot vol, e.g., Ledoit, Santa-Clara, and Yan (2002), Medvedev and Scaillet (2007, RFS), Henry-Labordere (2008), Aït-Sahalia, Li, and Li (2017)
- The idea behind is asymptotic expansion (AE):

$$I(m, \tau, \theta; \sigma_t) \sim \sigma_t + I_1(m, \theta; \sigma_t)\sqrt{\tau} + I_2(m, \theta; \sigma_t)\tau + \dots$$

  where $m = \frac{\log(X_t e^{(r-\delta)\tau}/K)}{\sigma_t\sqrt{\tau}}$, t is fixed, and $\tau \to 0$.
- AE ($\tau \to 0$ with a fixed $\#$)
  $\neq$ Taylor expansion ($\# \to \infty$, for a fixed τ)
- Motivation of AE is fast evaluation/calibration of option prices.
- Motivation of this paper is to *estimate* σ_t^2 *nonparametrically* using *many* such options.

Inference using Approximated Prices

- Consistency requires uniform approximation over parameter space

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^{n}L(\theta;X_i)-\mathrm{E}L(\theta;X)\right|\xrightarrow{p}0\implies\widehat{\theta}\xrightarrow{p}\theta_0$$

- Taylor approximation needs to satisfy

$$\sup_{\theta\in\Theta}\left|\tilde{L}(\theta,\#;X_i)-L(\theta;X_i)\right|\xrightarrow{a.s.}0,\quad \text{as}\quad \#\to\infty,\quad \forall\tau\leq\overline{\tau}$$

  where $\#\to\infty$ is not associated with $n\to\infty$.

- Asymptotic expansion needs to satisfy

$$\sup_{\theta\in\Theta}\left|\tilde{L}(\theta,\tau;X_i)-L(\theta;X_i)\right|\xrightarrow{a.s.}0,\quad \text{as}\quad \tau\to 0$$

  In the limit we have no data.

- In the Merton's jump diffusion case, I show that the asymptotic expansion for option prices is a convergent series: Xiu (2014, JoE), see also Lemma 2 of this paper:

$$\left| \widetilde{O}_\tau(k) - (e^k - 1)\Phi\left(\frac{k}{\sqrt{\tau}\sigma_t}\right) - \sigma_t\sqrt{\tau}\phi\left(\frac{k}{\sqrt{\tau}\sigma_t}\right) \right| \leq C\tau, \forall \tau \leq \bar{\tau}$$

- For general Lévy processes, the leading term should be dominated by jumps.
- For stochastic volatility processes, the leading term should not be Gaussian pdf or cdf, depending instead on heat kernels.
- This paper manages to get around both problems using Lemma 1, taken from another paper by Qin and Todorov (2017).

Asymptotic Analysis

- Asymptotic analysis (in statistics) relies on *sample size* $\to \infty$.
  - cross-section or time series data, panel data, high frequency data, etc
- In this paper, the usual sequences are

$$\overline{K} \to \infty, \quad \underline{K} \to 0, \quad \overline{\Delta} \to 0,$$

which are related to the expansion of a finer strike space, hence leading to more data.

Asymptotic Analysis

- However, asymptotic analysis in this paper demands a joint convergence among $\overline{K}$, $\underline{K}$, $\overline{\Delta}$, and τ, such that as $\tau \to 0$, sample size increases.
- The thought experiment is a bit unusual – "In order to accommodate 100 more data, we need to use options with 1-week shorter maturities."
- Not to mention in the limit, there are no data left. In finite sample, it is possible that we have no data available when the CLT finally kicks in.

Final Remark

- The weakness of this paper, if any, lies in the empirical appeal of the estimator.
- However, the technical side is very interesting, which has already led to some exciting applications, e.g., Torben, Fusari, and Todorov (JF, 2017).