Π-CAPM: The Classical CAPM with Probability Weighting and Skewed Assets

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Abstract

We study asset prices in a generalized mean-variance framework that allows for probability weighting (the idea that investors overweight rare, high impact events). The resulting model—the Π-CAPM—allows for a unique and homogeneous pricing equilibrium with skewed and correlated assets and a tractable analysis thereof. We find that even symmetric probability weighting has asymmetric pricing implications. For example, the price impact of volatility is skewness-dependent, negative for left-skewed assets but potentially positive for right-skewed assets. We further find that probability weighting translates into an exaggerated dependence between the assets. Finally, we make an empirical contribution and show that the option-implied premiums on variance and skewness depend on the underlying asset's skewness, in the very way that is predicted by the Π-CAPM.

Keywords: Asset pricing, behavioral finance, probability weighting, option markets.

JEL codes: G02, G11, G12.

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1 Introduction

Research from psychology has shown that humans often overweight the probability of rare, high impact events—such as those of unlikely but extreme financial returns. To take such probability weighting into account in the human evaluation of risks, Edwards (1962) proposed the replacement of objective probabilities with decision weights. This idea was formalized in the rank-dependent utility model of Quiggin (1982) and in 1992's (Tversky and Kahneman) prospect theory, among others, and aided economists in studying the implications of probability weighting in numerous economic and financial settings. In this paper, we study the implications of the most basic implication of probability weighting—overweighting of the tails of the distribution—for asset prices. The investors in our model have distorted mean-variance preferences (i.e., mean-variance preferences with probability weighting). We abbreviate these preferences as Π-MV, where—as is common in the behavioral economics literature—Π stands for the set decision weights (i.e., the distorted probabilities) that the investor uses instead of the objective probabilities when evaluating risky assets. The new model that is based on Π-MV preferences—the Π-CAPM—extends the classical CAPM of Lintner (1965) and Sharpe (1964) by a single parameter that captures probability weighting and reduces to it if that parameter is zero.

The Π-CAPM allows for the identification of the effect of tail overweighting on asset prices and its predictions compare directly to those of the classical CAPM. We assume a financial market that consists of two binary assets, which allows for simple and transparent comparative statics analysis with respect to the assets' volatility, skewness, and correlation. Moreover, binary assets ensure the absence of arbitrage opportunities for reasonable parameter values that otherwise may arise in a mean-variance framework (Dybvig and Ingersoll 1982). A crucial assumption on the financial market is that of no short-selling. As will be explained, short-selling interacts with the assumption of probability weighting in a non-trivial way that would make the analysis intractable. Probability weighting is modeled using the well-proven single-parameter neo-additive weighting function of Chateauneuf et al. (2007), which results in overweighting extreme good and extreme bad events symmetrically. These assumptions jointly ensure that the Π-CAPM largely retains the tractability of the classical CAPM. They allow for a number of analytical predictions—some known and some entirely new—regarding the pricing of skewed and correlated assets, as well as of options signed on them, in a standard (i.e., homogeneous holdings) pricing equilibrium.

The Π-CAPM makes a number of specific predictions on the pricing of variance and skewness in stocks and stock options. First, while the price of a left-skewed asset increases in skewness and decreases in the volatility of the asset, the price of a right-skewed asset may decrease in skewness and increase in volatility. Intuitively, under probability weighting, a right-skewed asset is generally overpriced (Barberis and Huang 2008), because the small probability of a large payoff is being overweighted. Increasing the volatility of such an asset increases the payoff that is received with this overweighted probability and may thus be desirable. That is, the price impact of volatility is skewness-dependent in the Π-CAPM. The intuition behind the result that increasing the skewness of a right-skewed asset may eventually decrease its price—the price impact of skewness is also skewness-dependent—is more subtle and explained in main part of the paper. Our equilibrium model with two skewed assets allows for another novel implication regarding the pricing of skewness that concerns how skewed the assets are compared to one another. Specifically, the Π-CAPM predicts that skewness of a given asset is more positively priced if the other asset is slightly left-skewed as compared to slightly right-skewed.

Second, investors with probability weighting exaggerate the dependence between the assets. Intuitively, probability weighting results in the overweighting of small-probability, extreme (low or high-payoff) events. The extreme events are those in which all assets do either well or badly. But, these are just the events in which the assets co-moving. Therefore, the probability of co-movement is overweighted. While this implication of probability weighting (exaggerating the dependence of assets) seems rather straightforward, to the best of our knowledge it has not been noted before. We show that it has consequences for the correlation premium that results from selling realized correlation (Driessen et al. 2009). In line with existing empirical evidence, the Π-CAPM predicts it to be positive and sizable, while with standard preferences it is small and may be even negative.

Third, two further novel predictions of the Π-CAPM concern the variance premium (Bollerslev et al. 2009; Carr and Wu 2009) and the skewness premium (Kozhan et al. 2013). The variance premium of the stock market is known to be positive, which means that the strike of a variance swap exceeds the market's realized variance; selling realized variance earns a risk premium. The Π-CAPM predicts that the variance and skewness premium of an individual asset, respectively, depend on its skewness in specific ways. The variance premium of a sufficiently asymmetric (i.e., sufficiently left- or right-skewed) individual stock is positive, and increasing in the stock's asymmetry. The skewness premium increases in the skewness of the underlying stock, is negative for left-skewed stocks, and is positive for right-skewed stocks. In the empirical part of the paper, we verify these predictions regarding skewness dependence using the cross-section of U.S. individual stock options. To this end, we adopt (and slightly adapt) the methodology of Kozhan et al. (2013) that was developed for stock market index options. In particular the results that the variance and skewness premiums of right-skewed stocks are positive are noteworthy, because they are difficult to explain with standard preferences (as we also show).

Related literature. This paper contributes to several strands of literature. First, this paper contributes to the theoretical asset pricing literature by studying the implications of a novel asset pricing model with probability weighting. As probability weighting complicates equilibrium analysis substantially, there are relatively few models that incorporate it, and they come with different assumptions, solution concepts, and areas of focus in terms of application. Barberis and Huang (2008) were the first to show that, with probability weighting, an individual asset's own skewness is priced. This result is obtained within a heterogeneous holdings equilibrium in which investors with identical preferences may optimally hold different portfolios. Polkovnichenko and Zhao (2013) show that probability weighting can explain non-monotone pricing kernels. Baele et al. (2019) focus on a single-asset economy and show that probability weighting simultaneously explains the cross-section of put and call option prices as well as the positive variance premium. Barberis et al. (2020) propose and calibrate a dynamic model that includes all aspects of prospect theory (including probability weighting), and solve it numerically to examine its ability to explain 23 prominent stock market anomalies. Some of the distinguishing aspects of our model are that it nests and extends the classical Lintner-Sharpe CAPM and allows for homogeneous pricing equilibrium with several correlated assets and multiple investors. It allows for making analytical predictions, regarding the price impact of volatility and skewness (both being skewness-dependent), regarding the exaggerated dependence between the assets, and regarding option-implied risk premiums.

Second, the paper contributes to the theoretical literature on the behavioral implications of probability weighting more generally. Theoretical work has pointed to the importance of probability weighting for portfolio choice (Bernard and Ghossub 2010), insurance behavior (Bernard et al. 2015), and gambling decisions (Barberis 2012), among many others. More recently,empirical researchers have documented direct links between probability weighting and behavior.[1]

Third, the Π-CAPM theoretically underpins existing empirical work. For example, a number of authors have presented evidence in support of the prediction that skewness is negatively priced across many asset classes, including stocks and options, for example Kumar (2009), Boyer et al. (2010), Bali et al. (2011), Conrad et al. (2013), Boyer and Vorkink (2014), Ghysels et al. (2016), and Schneider (2019). As individual stock returns tend to be right-skewed, the Π-CAPM's prediction that the returns of right-skewed assets typically decrease in volatility offers an explanation of the idiosyncratic volatility puzzle as described in Ang et al. (2006). It makes the more refined prediction that, in the cross-section of stock returns, the idiosyncratic volatility puzzle should not be observed for left-skewed stocks. Barberis et al. (2020) explain the low returns on stocks with high volatility by the fact that these stocks have, on average, higher skewness and worse past performance. The Π-CAPM's prediction that right-skewed assets earn (even) lower expected returns when the remainder of the market is left-skewed is consistent with the results of Gao and Lin (2015), who show that trading volume in Taiwanese lottery-like stocks and the lottery jackpot are negatively correlated. In an experimental setting, Dertwinkel-Kalt and Köster (forthcoming) have shown that choices are affected by how skewed two risks appear relative to each other. The Π-CAPM predicts a sizable correlation premium, and predicts that this premium decreases in the objective correlation between the assets. The former prediction is documented by Driessen et al. (2009) and Buss et al. (2016) in the equity market, and by Mueller et al. (2017) in the foreign

[1]Barseghyan et al. (2013) find that probability weighting can explain insurance deductible choices. Barberis et al. (2016) find that stocks with return distributions that are appealing under probability weighting underperform subsequently. Dimmock et al. (forthcoming) obtain survey evidence that probability weighting is positively associated with portfolio under diversification and Sharpe ratio losses. Moreover, higher probability weighting is associated with owning lottery-like stocks and positively skewed equity portfolios.

exchange market. The latter prediction is in line with Buss et al. (2016) who show that the correlation premium of an industry decreases in the objective correlation of the stocks within that industry, and in line with Mueller et al. (2017) who show that the correlation premium in the foreign exchange market decreases in the objective correlation between the currencies.

Finally, we make an empirical contribution to the literature on the option-implied variance and skewness premiums. The stylized fact that the variance premium of the stock market is positive is well documented, as shown by Bollerslev et al. (2009), Carr and Wu (2009), Kozhan et al. (2013), Dew-Becker et al. (2017), and Baele et al. (2019). For individual stocks, however, the evidence is somewhat mixed. As Carr and Wu (2009) and Driessen et al. (2009) show, for individual stocks there is a lot of cross-sectional variation in the sign, as well as in the size, of the variance premium. Our main contribution is to show that the skewness of the underlying distribution is an important determinant of the sign and the size of the variance premium, in the very way that is predicted by the Π-CAPM. As regards the skewness premium, Kozhan et al. (2013) show that the skewness premium of the stock market is negative and strongly related to the variance premium. We replicate their result using a slightly different methodology and for an extended time period. To the best of our knowledge, we are the first to study the skewness premiums of individual stocks.[2]

2 Model

In this section, we define the financial market and the investor's preferences. Afterward, we determine the equilibrium prices of the assets in the economy and determine the no-arbitrage condition. The section closes with a calibration of the model that will be used to illustrate some of its implications discussed in Section 3.

[2]There are papers, however, that utilize skewness implied from individual options to study expected return variation in the cross-section of stock returns, such as Xing et al. (2010), Conrad et al. (2013), and Schneider et al. (2020).

2.1 Financial Market

The market we consider consists of two binary assets, X and Y. Letting $p_X \in (0,1)$ and $\bar{x} > \underline{x}$, asset X pays $\bar{x}$ with probability p_X and $\underline{x}$ otherwise. The payoff distribution of asset Y is defined analogously; see Figure 1.

Figure 1: Asset X and Y.

Binary distributed assets bring tractability and allow for effective comparative statics analyses with respect to the moments of the distribution. In particular, mean, variance, and (standardized) skewness of asset X are given by:

$$\begin{aligned}
\mu_X &:= \mathbb{E}X = p_X(\bar{x} - \underline{x}) + \underline{x}, \\
\sigma_X^2 &:= \mathbb{E}\big(X - \mu_X\big)^2 = p_X(1-p_X)(\bar{x} - \underline{x})^2, \\
S_X &:= \frac{\mathbb{E}\big(X - \mu_X\big)^3}{\sigma_X^3} = \frac{1 - 2p_X}{\sqrt{p_X(1-p_X)}},
\end{aligned}$$

and analogously for asset Y. First, note that, even though the binary distribution is very simple, it can independently match the first *three* moments of the payoff distribution. A lognormal distribution, for example, while maybe appearing to be more general due its continuous nature, matches only two moments (typically mean and variance). The value of skewness is redundant and described by a complex function of the mean and standard deviation parameters. Moreover, a lognormal distribution requires skewness to be positive; that is, it is not possible to study negative skewness with a lognormal distribution. A binary distribution, in contrast, can match any mean and skewness value, and any strictly positive variance value.[3] The assumption of binary payoff distributions

[3]Formally, there exists a one-to-one mapping between the parameter triples $(p_X, \bar{x}, \underline{x})$ and (μ_X, σ_X^2, S_X); see Proposition 1 in Ebert (2015). Therefore, rather than parametrizing the distribution by two payoffs and probability, it can be parametrized by its first three moments, with any combination of values being feasible (except for variance being negative, of course). Moreover, it can be shown that all higher-order even (odd) moments are one-to-one to variance (skewness). That is, in the case of binary assets, the second (third) moment exhaustively describes the symmetric (asymmetric) nature of risk.

thus allows for a simple, transparent, and intuitive approach to the study of asset prices and their dependence on variance and skewness.[4]

The joint distribution of assets X and Y can be expressed as

$$\begin{aligned}
p_{\bar{x}\bar{y}} &:= \mathbb{P}\left(X = \bar{x}, Y = \bar{y}\right) = p_X p_Y + r_{XY}, \\
p_{\bar{x}\underline{y}} &:= \mathbb{P}\left(X = \bar{x}, Y = \underline{y}\right) = p_X(1 - p_Y) - r_{XY}, \\
p_{\underline{x}\bar{y}} &:= \mathbb{P}\left(X = \underline{x}, Y = \bar{y}\right) = (1 - p_X)p_Y - r_{XY}, \text{ and} \\
p_{\underline{x}\underline{y}} &:= \mathbb{P}\left(X = \underline{x}, Y = \underline{y}\right) = (1 - p_X)(1 - p_Y) + r_{XY},
\end{aligned}$$

where the parameter r_{XY} must be such that all state probabilities lie strictly between zero and one.[5] Intuitively, if the probability of both assets paying the higher amount exceeds the product of their marginal probabilities, the assets are co-moving. For the Pearson correlation coefficient $\rho_{XY} \equiv \operatorname{corr}(X, Y)$ it follows that

$$\rho_{XY} = \frac{r_{XY}}{\sqrt{p_X(1 - p_X)}\sqrt{p_Y(1 - p_Y)}} > 0 \iff r_{XY} > 0,$$

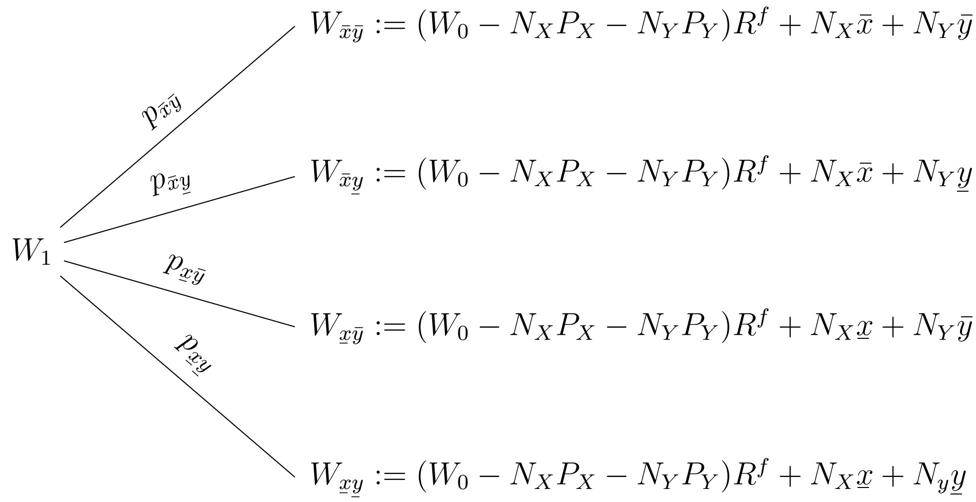
and r_{XY} can be chosen such that correlation stays constant as one considers changes in the skewness parameters p_X and p_Y. In summary, the bivariate binary distribution allows for ceteris paribus analysis regarding the pricing impact of all univariate moments as well as of their dependence in the form of correlation. For example, it will allow us to study the change in price of asset X as S_X changes while leaving all over univariate moments of X and Y as well as the correlation between them unchanged.

Consider an investor with initial wealth W_0. Denote her holdings (her demand) of assets X and Y, given prices P_X and P_Y, by N_X and N_Y, respectively. The remainder of her endowment is invested in a riskless asset with total return R^f. Then, her terminal wealth W_1 in the market's four states is given by the respective expressions in Figure 2.

[4]Binary distributions, in particular with the objective of studying skewness, are also used in other asset pricing settings (e.g., Barberis and Huang (2008) and Schneider (2015)). Beddock and Karehnke (2020) study an asset pricing setting with two skewed-normal risks.

[5]r_{XY} must lie within the so-called Frechét bounds shown in equations (32). Allowing for state probabilities to be exactly zero is possible, but comes with technical inconvenience and case distinctions in the proofs, with little gain in the way of economic insight.

Figure 2: Terminal wealth.



Prices and demand are determined in equilibrium and depend on the investor's preferences over W_1.

2.2 Preferences

We assume a representative investor with distorted mean-variance preferences over wealth in period one.[6] In the same way as in the rank-dependent utility (RDU) model of Quiggin (1982) and in cumulative prospect theory (CPT, Tversky and Kahneman 1992), the investor uses decision weights (i.e., distorted probabilities) instead of the objective probabilities in evaluating risks. As will be explained, the direct effect of this probability distortion will be an overweighting of the tails of the wealth distribution. Specifically, with reference to Figure 2, given decision weights π_{ij} of the states ij with $i \in \{\bar{x}, \underline{x}\}$ and $j \in \{\bar{y}, \underline{y}\}$, the representative investor with risk-aversion parameter γ evaluates W_1 as:

$$U(W_1) = \mathbb{E}^{\Pi}(W_1) - \frac{1}{2}\gamma \mathrm{var}^{\Pi}(W_1), \tag{1}$$

where

$$\mathbb{E}^{\Pi}(W_1) = \sum_{i,j} \pi_{ij} \cdot W_{ij} \qquad \text{and} \qquad \mathrm{var}^{\Pi}(W_1) = \mathbb{E}^{\Pi}\big(W_1 - \mathbb{E}^{\Pi} W_1\big)^2.$$

[6]Under the subsequently made assumptions, we could start out with multiple investors who aggregate to a representative investor; see Section 2.3 for details. Ingersoll (2014) shows that such aggregation is not generally guaranteed with probability weighting.

Before we detail the computation of the decision weights π_{ij}, we make four remarks on the use of Π-MV preferences in general. First, when $\pi_{ij} = p_{ij}$ for each state ij, U describes standard mean-variance (MV) preferences. Therefore, as we will demonstrate explicitly below, our asset pricing model nests the standard CAPM of Sharpe (1964) and Lintner (1965) as a special case. Any differential pricing implications are thus due to the change from the p_{ij}'s to π_{ij}'s; that is, due to probability weighting. Everything else and, in particular, the equilibrium concept employed is standard. Different from CPT models such as Barberis and Huang (2008), our probability weighting model allows for the study of prices in a standard homogeneous equilibrium (in which investors with identical preferences hold the market portfolio).

Second, Π-MV preferences relate to RDU just as (standard) mean variance preferences relate to EU. Either can be obtained from a second-order Taylor expansion of utility. Π-MV preferences, however, are more tractable and allow for the analysis of skewed and correlated assets presented in this paper. The tractability allows us to obtain a number of predictions analytically.

Third, another natural extension of MV preferences are mean-variance-skewness (MVS) preferences. Models based on MVS preferences make quite different pricing predictions, both conceptually and quantitatively. While Π-MV preferences have implications different from MV preferences even when risks are symmetric, MVS preferences do not. For example, MVS preferences may predict a negative correlation risk premium in this case while Π-MV preferences predict a positive one (as is in line with the data). Further, it can be verified that only Π-MV preferences feature the sizable "first-order" preference effects regarding skewness that are known for models with probability weighting (Ebert and Karehnke 2019).

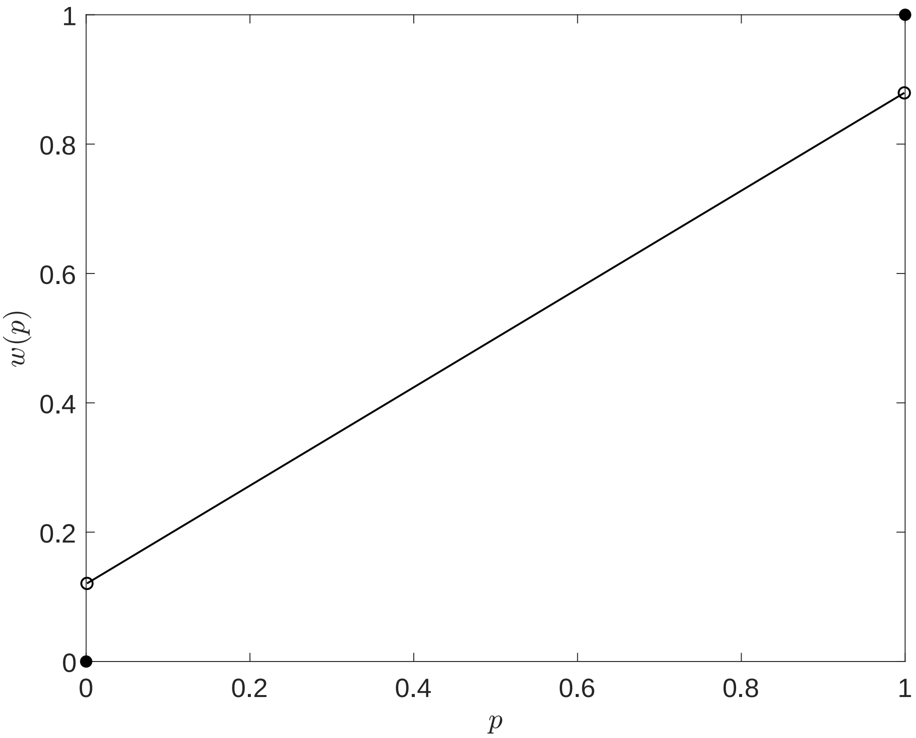
Fourth, because MV preferences are not in general monotonic, their application in asset pricing models may violate no-arbitrage (Dybvig and Ingersoll 1982). In Section 2.4, we derive the necessary and sufficient condition for no-arbitrage in the Π-CAPM and show that the bounded support of the binary assets averts arbitrage for reasonable parameter values.

As in RDU and CPT, the decision weights π_{ij} are computed by means of a so-called probability weighting function. We assume the probability weighting function to be the single-parameter neo-additive function of Chateauneuf et al. (2007).[7] For $a \in \left[0, \frac{1}{2}\right)$ it is defined as:

$$w(p) = \begin{cases} 0 & \text{for} \quad p = 0, \\ (1-2a)p + a & \text{for} \quad 0 < p < 1, \\ 1 & \text{for} \quad p = 1. \end{cases} \tag{2}$$

A graph of the neo-additive probability weighting function is shown in Figure 3.

Figure 3: The neo-additive weighting function.
This figure shows the neo-additive weighting function given by equation (2) when $a = 0.12$. It is linear in the interior of its domain and discontinuous at probabilities zero and one.



Decision weights are computed as differences of the weighted cumulative probabilities of the ranked outcomes. Assuming (for now) that this ranking is given by

[7]The neo-additive weighting function belongs to the class of inverse-S shaped weighting functions and yields, similar to the weighting function by Kahneman and Tversky (1979), the economic prediction of tail overweighting. Unlike the latter weighting function and others, the effect of tail overweighting is not conflated with other, subtler, and sometimes unintended effects that stem from the specific parametric form assumed. Wakker (2010, p. 210) remarks that "the neo-additive weighting functions are among the most promising candidates regarding the optimal tradeoff of parsimony and fit" and that "the interpretation of its parameters is clearer and more convincing that with other families."

$W_{\bar{x}\bar{y}} \geq W_{\bar{x}\underline{y}} \geq W_{\underline{x}\bar{y}} \geq W_{\underline{x}\underline{y}}$, they can be computed analytically:

$$\Pi = \begin{cases} \pi_{\bar{x}\bar{y}} = w\big(p_{\bar{x}\bar{y}}\big) & = a + (1-2a)p_{\bar{x}\bar{y}} \\ \pi_{\bar{x}\underline{y}} = w\big(p_{\bar{x}\bar{y}} + p_{\bar{x}\underline{y}}\big) - w\big(p_{\bar{x}\bar{y}}\big) & = (1-2a)p_{\bar{x}\underline{y}} \\ \pi_{\underline{x}\bar{y}} = w\big(p_{\bar{x}\bar{y}} + p_{\bar{x}\underline{y}} + p_{\underline{x}\bar{y}}\big) - w\big(p_{\bar{x}\bar{y}} + p_{\bar{x}\underline{y}}\big) & = (1-2a)p_{\underline{x}\bar{y}} \\ \pi_{\underline{x}\underline{y}} = 1 - w\big(p_{\bar{x}\bar{y}} + p_{\bar{x}\underline{y}} + p_{\underline{x}\bar{y}}\big) & = a + (1-2a)p_{\underline{x}\underline{y}}. \end{cases} \tag{3}$$

Note that the decision weights are non-negative and sum to one so that Π describes a probability measure. The common interpretation of Π is, however, one in terms of preferences—the investor prefers to weigh states differently than by their objective probabilities—rather than one in terms of having false beliefs about those objective probabilities; see Kahneman and Tversky (1979) and Quiggin (1982).

We now explain the implications of neo-additive probability weighting for preferences. When $a = 0$, $w(p) = p$ so that decision weights and objective probabilities coincide and preferences are standard MV. When $a > 0$, the investor distorts probabilities. The direct effect of this probability distortion is the overweighting of the tails of the wealth distribution—in the current case, of the best state $\bar{x}\bar{y}$ and the worst state $\underline{x}\underline{y}$ (the two "extreme states"). Specifically, a fraction of $2a$ is removed from the probability mass of all four states, and the resulting total of $2a(p_{\bar{x}\bar{y}} + p_{\bar{x}\underline{y}} + p_{\underline{x}\bar{y}} + p_{\underline{x}\underline{y}}) = 2a$ is, in equal proportions, redistributed to only the two extreme states.

Why do we choose the neo-additive weighting function to compute distorted probabilities? As will be explained in the next section, this weighting function buys us tractability. Moreover, recent theoretical research has argued in favor of the neo-additive weighting function; see, for example, Wakker (2010 p. 209-210). As explained in the previous paragraph, the parameter a has a simple and unambiguous interpretation in terms of tail overweighting. While the parameter of the original Kahneman-Tversky weighting function, for example, is typically also interpreted in terms of tail overweighting, this interpretation is confounded with subtle other effects that are more difficult to interpret as well as assessed quantitatively. In particular, our assumption of the point-symmetric (one-parameter) neo-additive weighting function ensures that the redistribution of probability mass is not done in favor of either tail. A major point of this paper is that, even though for $a = 0$ preferences are MV so that skewness is irrelevant, and even though $a > 0$ does nothing else other than symmetrically overweighting the tails, $a > 0$ yields asymmetric pricing predictions for left- and right-skewed risks. In this first study on distorted MV preferences we do not wish to confound this asymmetric pricing effect of symmetric probability distortion with any additional effects that result from probability distortion being asymmetric in itself.

Before we derive and discuss the equilibrium of the model, we impose the standing assumption that $p_Y = 0.50$ (i.e., asset Y symmetric). This assumption mirrors the normality assumption in Barberis and Huang (2008). Another reason for this assumption is that we are mostly interested in the pricing effects of an asset's own skewness. However, in Section 3.6, we relax this assumption and discuss the pricing effect of relative skewness.

Standing assumption. Asset Y is symmetric ($p_Y = 0.50$).

2.3 Equilibrium Prices

In this section, we derive the equilibrium prices of the assets. Inserting the expression for terminal wealth W_1 in Figure 2 into the value function (1) yields:

$$\begin{aligned} U(W_1) = &W_0 R^f + N_X\big(\mathbb{E}^\Pi X - P_X R^f\big) + N_Y\big(\mathbb{E}^\Pi Y - P_Y R^f\big) \\ &- \frac{1}{2}\gamma N_X^2 \mathrm{var}^\Pi\big(X\big) - \frac{1}{2}\gamma N_Y^2 \mathrm{var}^\Pi\big(Y\big) - \gamma N_X N_Y \mathrm{cov}^\Pi\big(X, Y\big). \end{aligned} \tag{4}$$

In equilibrium, prices are such that the asset demands N_X and N_Y maximize $U(W_1)$ and equal their respective supplies $\bar{N}_X > 0$ and $\bar{N}_Y > 0$. To ensure optimality of demand in equilibrium, we assume that the investor can only take long positions in either asset (i.e., demand is non-negative). Theorem 1 below shows that an equilibrium exists and presents the closed-form pricing equations of our model.

Before we state and discuss the equilibrium described in Theorem 1, we highlight how our model overcomes a major difficulty in equilibrium models with probability distortion. As emphasized in the sentence preceding the expressions for the decision weights in equation (3), these expressions are contingent on $W_{\bar{x}\bar{y}} \geq W_{\bar{x}\underline{y}} \geq W_{\underline{x}\bar{y}} \geq W_{\underline{x}\underline{y}}$ (i.e., on "the ranking" of the wealth states). Since demand and asset prices are endogenous in equilibrium analysis, so is terminal wealth in each state and the ranking. The issue is that a small/continuous change in the wealth of two states can change their ranking, which, in general, leads to a large/discontinuous change of their decision weights. Our model circumvents this issue. First, due to the assumption of no short-selling, the highest and lowest ranked states are always—that is, no matter the demand of each asset—$\bar{x}\bar{y}$ and $\underline{x}\underline{y}$, respectively. Second, due to the linearity of the neo-additive probability weighting function on the interior of its domain, the decision weights of the two middle states do not depend on their ranking (i.e., on whether $W_{\bar{x}\underline{y}} \geq W_{\underline{x}\bar{y}}$ or vice versa).[8] Therefore, no matter the demand of each asset, the decision weights are given by the right-hand-side expressions in equation (3). This is a unique feature of the Π-CAPM that distinguishes it from other asset pricing models with probability weighting. Elementary methods then allow for the derivation of homogeneous equilibrium, as with standard MV preferences. The proofs of the following theorems and propositions can be found in the appendix of the paper.

Theorem 1 (Equilibrium Prices in the Π-CAPM)**.** *In equilibrium, the pricing equations are given by:*

$$P_X R^f = \mathbb{E}^{\Pi} X - \gamma \bar{N}_X \mathrm{var}^{\Pi}(X) - \gamma \bar{N}_Y \mathrm{cov}^{\Pi}(X, Y), \tag{5}$$

$$P_Y R^f = \mathbb{E}^{\Pi} Y - \gamma \bar{N}_Y \mathrm{var}^{\Pi}(Y) - \gamma \bar{N}_X \mathrm{cov}^{\Pi}(X, Y). \tag{6}$$

Equations (5) *and* (6) *can be rewritten in terms of non-distorted moments, as follows:*

$$\begin{aligned} P_X R^f = \mu_X + a S_X \sigma_X - \gamma \bar{N}_X \left[\sigma_X^2 + a(1-a)\sigma_X^2 S_X^2\right] \\ - \gamma \bar{N}_Y \left[\mathrm{cov}(X, Y)(1-2a) + a\frac{\sigma_X \sigma_Y}{\sqrt{p_X(1-p_X)}}\right], \end{aligned} \tag{7}$$

$$P_Y R^f = \mu_Y - \gamma \bar{N}_Y \sigma_Y^2 - \gamma \bar{N}_X \left[\mathrm{cov}(X, Y)(1-2a) + a\frac{\sigma_X \sigma_Y}{\sqrt{p_X(1-p_X)}}\right]. \tag{8}$$

[8] It is also for this reason (i.e., the decision weights of middle states being independent of their ranking) that the model is easily extended to a finite number of assets. More middle states do not complicate the analysis if probability weighting is neo-additive.

Theorem 1 states that the prices of the assets correspond to a linear combination of distorted mean, distorted variance, and distorted covariance. In the proof of Theorem 1, we obtain the following equations for the distorted moments of asset X in order to derive equation (7):

$$\mathbb{E}^{\Pi} X = \mu_X + a S_X \sigma_X, \tag{9}$$

$$\mathrm{var}^{\Pi}(X) = \sigma_X^2 + a(1-a) S_X^2 \sigma_X^2, \tag{10}$$

$$\mathrm{cov}^{\Pi}(X,Y) = (1-2a)\mathrm{cov}(X,Y) + a \frac{\sigma_X \sigma_Y}{\sqrt{p_X(1-p_X)}}. \tag{11}$$

These equations offer important first insight into the price comparative statics of our model, for example, regarding the impact of asset X's skewness, S_X. Equation (9) shows that the investor evaluates the mean of the asset differently from its objective mean. Specifically, the asset's distorted mean is an affine function of S_X and larger than the objective mean if and only if skewness is positive. Equation (10) shows that the distorted variance is always larger than its objective counterpart and increasing in the asset's *asymmetry* (i.e., the absolute, or the square, of skewness). Similarly, recalling the negative one-to-one relationship between p_X and S_X above, Equation (11) implies that also the distorted covariance increases in the asset's asymmetry. In all three cases, the effects increase in the distortion parameter a. In Section 3.1, we study the joint impact of these distortion effects on prices and see that they go sometimes in the same and sometimes in opposite directions. One implication is that an increase in skewness mostly increases the price of the asset (for example, when it is left-skewed and when all three effects go in the same direction), but, in other cases, may also decrease it.

Theorem 1 presents the pricing equations of the assets in equilibrium where the investor evaluates risk according to probability weighting. Note that, in case $a = 0$ pricing equation (7) can be rewritten as:

$$P_X R^f = \mu_X - \gamma \cdot \mathrm{cov}\big(X, \bar{N}_X X + \bar{N}_Y Y\big),$$

which says that P_X is determined by its mean minus the price of risk (γ) multiplied with the covariance between its payoff and the market portfolio. Then, the standard CAPM equation (Sharpe 1964; Lintner 1965), in terms of returns, is easily derived:

$$\mathbb{E}R_X - R^f = \beta_X \mathbb{E}\big(R_m - R^f\big), \tag{12}$$

where,

$$R_X = \frac{X}{P_X},\ R_m = \frac{\bar{N}_X X + \bar{N}_Y Y}{\bar{N}_X P_X + \bar{N}_Y P_Y},\ \text{and } \beta_X = \frac{\mathrm{cov}(R_X, R_m)}{\mathrm{var}(R_m)}.$$

With probability distortion, equation (12) no longer holds,[9] and the novel pricing illustrations are more conveniently discussed in terms of prices, using the equations in Proposition 1.

The pricing equations of Theorem 1 are derived for an economy in which both assets are in strictly positive supply. In a *homogeneous equilibrium,* the representative investor optimally holds the market portfolio (i.e., both assets). While, for simplicity, we assumed a representative investor, the same equilibrium is obtained under the assumption of many, identical investors. In the asset pricing model of Barberis and Huang (2008), probability weighting leads to a different type of equilibrium in an economy with many investors and in which the skewed asset is in infinitesimal supply (and thus not part of the market portfolio). In this *heterogeneous equilibrium,* different investors hold different portfolios. Some investors only hold the market portfolio while others hold the market portfolio and the small skewed asset. In this equilibrium, the skewed asset earns a low expected return. In case a homogeneous equilibrium exists, Barberis and Huang (2008) show that the expected return of the skewed asset is equal to the risk-free rate. Different from this result of Barberis and Huang (2008), Theorem 1 shows that, in our homogeneous equilibrium, the price of the skewed asset is (always) affected by its skewness, and the expected return will not be equal to the risk-free rate. In particular, while negatively skewed assets earn the risk-free rate in the homogeneous equilibrium of Barberis and Huang (2008), our model predicts substantial positive returns for negatively skewed assets.

[9] In this sense, the Π-CAPM is itself not a CAPM. It is a model that results from distorting the classical CAPM.

2.4 No-arbitrage and risk-neutral pricing measure

In Theorem 2, we characterize no-arbitrage in the Π-CAPM and derive the risk-neutral pricing measure in closed form.

Theorem 2 (No-arbitrage in the Π-CAPM)**.** *The following statements are equivalent:*

1. *The* Π*-CAPM is arbitrage free with unique risk-neutral measure* $\mathbb{Q}$*, defined through the state probabilities*

$$q_{ij} = p_{ij} \cdot \frac{\pi_{ij}}{p_{ij}} \Big[1 - \gamma \bar{N}_X (i - \mu_X - a\sigma_X S_X) - \gamma \bar{N}_Y (j - \mu_Y) \Big], i \in \{\bar{x}, \underline{x}\}, j \in \{\bar{y}, \underline{y}\}. \quad (13)$$

2. *The* Π*-CAPM model parameters satisfy*

$$\frac{1}{\gamma} > \bar{N}_Y \sigma_Y + 2\sqrt{a(1-a)}\bar{N}_X \sigma_X \quad (14)$$

and $p_X \in (p_1^*, p_2^*)$ *for* $0 < p_1^* < p_2^* \leq 1$ *whose expressions are stated in the proof.*

Equations (13) in statement 1 describe the pricing kernel of the Π-CAPM. The factor π_{ij}/p_{ij} describes the inflation or deflation of the objective probability p_{ij} that is due to probability weighting. Extreme states, be they good or bad, receive more weight while the middle states receive less weight (see above). The extreme good state is relatively more overweighted than the bad state ($\pi_{\bar{x}\bar{y}}/p_{\bar{x}\bar{y}}) > \pi_{\underline{x}\underline{y}}/p_{\underline{x}\underline{y}}$) if and only if its objective probability is smaller (i.e., if X is right-skewed). The term in the square brackets corresponds to what would be marginal utility in an EU model. Under MV preferences, payoffs enter the euqations relative to the mean payoff, while under Π-MV preferences these payoffs enter relative to the distorted mean payoff; see equation (9). In either case, marginal utility is relatively lower in states with high wealth compared to the (distorted) mean. Recall that the term $a\sigma_X S_X$ describes the difference between distorted and objective mean so that the sign of the asset's skewness determines whether the distorted mean is smaller (if $S_X < 0$) or larger (if $S_X > 0$) than its objective counterpart. No-arbitrage requires that all q_{ij} are positive or, equivalently, that Π-MV preferences are monotone over the relevant parameter range.

The no-arbitrage condition in statement 2 ensures that the assets' payoffs are such that the Π-MV preferences are monotone for the relevant risks considered. This is possible because our assets have finite support to begin with. If one restricts the model parameters appropriately, arbitrage opportunities as in classical articles on models with mean-variance preferences such as Dybvig and Ingersoll (1982) do not arise. We discuss no-arbitrage in some more detail in Appendix 7.11. There we show that, as asset X or risk aversion becomes small (technically, $\gamma \bar{N}_X \sigma_X^2$ close to zero), no-arbitrage holds for *any* level of skewness (technically, (p_1^*, p_2^*) close to $(0,1)$). The model calibration outlined below satisfies no-arbitrage for empirically realistic skewness values and beyond. Likewise, all of our propositions make implications for arbitrage-free versions of the Π-CAPM.[10]

2.5 Calibration of the Model

In this section, we discuss the calibration of the model and explain the parameter choices. This calibration serves to illustrate the economic significance of the analytical results we prove in the next section. Throughout the paper, we use the calibration of the financial market and preference parameters of Table 1.

Table 1: Model Calibration.
The table shows the parameter choices for mean μ_X, volatility σ_X, skewness S_X, correlation ρ_{XY}, supply of asset X $\bar{N}_X$, probability distortion parameter a, and risk-aversion parameter γ. The parameters for asset Y are denoted analogously.

Parameter	Value	Parameter	Value
μ_X	1	ρ_{XY}	0
μ_Y	1	$\bar{N}_X$	$\frac{1}{10}$
σ_X	0.20	$\bar{N}_Y$	1
σ_Y	0.20	R^f	1.01
S_X	$\big[-4.13, 4.13\big]$	a	0.12
S_Y	0	γ	1.5

As shown in Ebert (2015), the distribution of each asset X and Y is characterized by its mean, standard deviation, and skewness of the payoff. The expected payoff of each

[10] In Appendix 7.11, we also compare no-arbitrage in the presence and in the absence of probability weighting. Probability weighting neither relaxes nor tightens the no-arbitrage conditions and modifies them only slightly.

asset is normalized to one. Due to this normalization, the standard deviation of the asset's payoff will be close to the standard deviation of the asset's return.[11] $S_Y = 0$ is in line with our standing assumption on the skewness of asset Y, and the skewness level for asset X is in line with skewness levels of the skewed asset in Barberis and Huang (2008) and empirical levels reported in Barberis et al. (2020). Lastly, we choose the correlation between the assets to be zero for simplicity. We normalize the supply $\bar{N}_Y$ to one and choose $\bar{N}_X = \frac{1}{10}$ such that a relatively small part of the market can be skewed. The risk aversion parameter $\gamma = 1.5$ is chosen such that the equity premium of the market portfolio is reasonable in the absence of probability distortion.[12] The probability distortion parameter a follows from Abdellaoui et al. (2010) by matching the slope of the weighting function to their calibration.[13] In case of probability weighting and when the skewness of both assets equals zero, the equity premium equals 6.2% with a volatility of 19.6%.

3 Predictions of the Π-CAPM

In this section, we present implications and predictions derived from the Π-CAPM. First, we explore the effects of increased skewness and variance of the skewed asset X on its price and its risk-neutral distribution. If asset X is left-skewed, its price is unambiguously increasing in its skewness and decreasing in its volatility. If asset X is right-skewed, however, these price effects depend on parameters. In particular, the price of asset X can be decreasing in its skewness and increasing in its volatility. We explain the mechanism underlying these skewness-dependent comparative statics results. Second, we show that the variance premium of an asset increases in the asymmetry (i.e., in the absolute of skewness) of its distribution, whereas the skewness premium increases with its skewness. In the last two subsections of this section, we focus on interaction effects of

[11] In fact, the standard deviation of the return for each asset is equal to the standard deviation of the asset's payoff divided by its price.

[12] In that case, the expected return on the market is unaffected by the skewness values of the assets. The equity premium equals 5.9% with a volatility of 19.5% in absence of probability weighting. It can be verified that the calibration implies the reasonable value of 3.10 for relative risk aversion for an investment of 50% in the stock market.

[13] Due to our assumption of symmetry in the weighting function, the chosen calibration follows immediately by matching the slopes of our weighting function with that of the estimate in Abdellaoui et al. (2010).

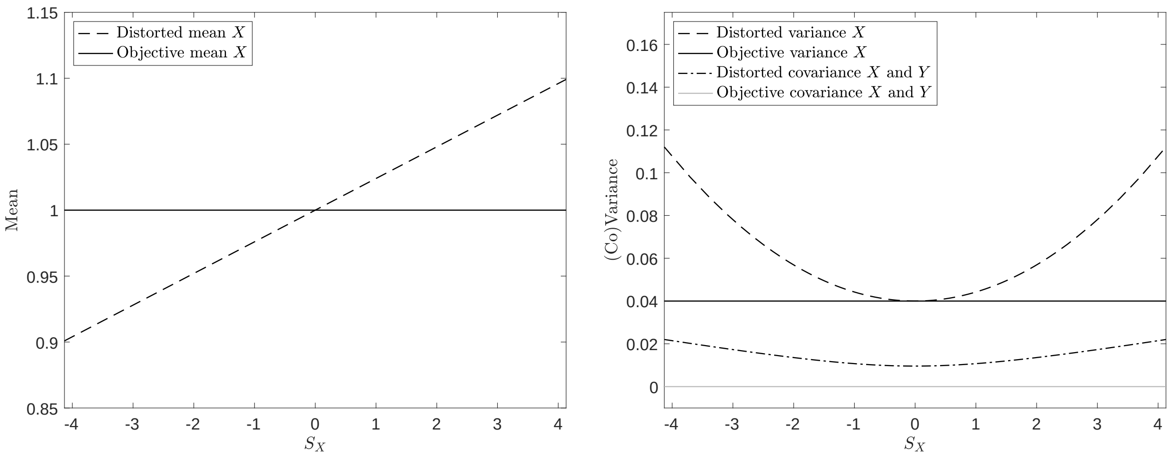
both the assets. We show that the model predicts a positive and economically significant correlation premium. Furthermore, we show that relative skewness matters for pricing: An increase in the skewness of asset X has less impact on its price when the skewness of asset Y larger.

Throughout this section, we relate the implications of our model to existing empirical work. Moreover, we illustrate them numerically alongside our model calibration from Section 2.5, which provides insight into the economic significance of our analytical results.

3.1 Skewness-dependent pricing of skewness

In Theorem 1, we have shown that the pricing equations of both assets depend on the skewness of asset X (S_X) due to its effect on the distorted mean, variance, and covariance; see also equations (9)—(11), respectively. Before we assess the pricing implications of skewness in general, we illustrate the effect of S_X on its distorted mean, variance, and covariance for our model calibration detailed in Table 1. This will help us in understanding the subsequent results better. In Figure 4, we plot the distorted mean, variance, and covariance of the assets in our benchmark calibration as a function of S_X.

Figure 4: Distorted mean, variance, and covariance of the assets.
This figure illustrates the effect of S_X on the distorted mean, distorted variance, and distorted covariance. The dashed lines represent the distorted mean and distorted variance of asset X. The solid lines represent the objective mean of asset X, objective variance of asset X and objective covariance between asset X and Y. The left graph plots the distorted mean as a function of S_X. The right graph plots the distorted variance of both assets as well as the distorted covariance as a function S_X.



In the left graph of Figure 4, the distorted (objective) mean of asset X is represented by the dashed (solid) line. The distorted mean of asset X is increasing in S_X. This result is driven by the fact that probability weighting makes the investor over-weight the probability of extreme events. If asset X is right-skewed, the small probability of an extremely good state is overweighted. This overweighting results in a distorted mean that is larger than the objective mean. If asset X is left-skewed, the reverse is true. Due to probability weighting, the probability of the bad state is overweighted, which yields a smaller distorted mean.

In the right graph of Figure 4, the distorted (objective) variance of asset X is represented by the dashed (solid) line, and the distorted (objective) covariance of the assets by the dash-dotted black (solid grey) line. The distorted variance of asset X increases in the asymmetry of X. The logic behind this result is similar to that for the distorted mean just discussed: Probability weighting makes the investor overweight the good state for a right-skewed asset and the bad state for a left-skewed asset, respectively. As the probability of the extreme state, either good or bad, is overweighted, distorted variance increases. The distorted covariance between X and Y is larger than the objective covariance for all S_X and increases in asymmetry of X. We prove this formally later on and refer to the result as "covariance exaggeration." Notably, probability weighting exaggerates covariance even when both assets are symmetric. Therefore, skewness is not the only channel through which probability weighting affects prices. One implication of exaggerated covariance is that the Π-CAPM predicts a positive correlation premium as will be discussed in Section 3.5.

After this preliminary discussion of the impact of skewness on the distorted moments that determine asset X's price, we now turn to the effect of skewness on asset prices. In Proposition 1, we formalize the effect of S_X on the price of asset X. Recall that our setting allows for varying the skewness of the asset's payoff, while maintaining its expected value, variance, and correlation with the other asset. The derivative in Proposition 1 below thus describes how price changes with respect to skewness while keeping any other moments constants; see Ebert (2015) for details. The same is true for later propositions that consider derivatives with respect to other moments—all of these derivatives describe ceteribus paribus moment changes, which is a major feature of the model setup in Section 2. Because, in particular, mean remains unchanged, the effect of

S_X on the price of X can also be directly interpreted in terms of X's expected return.

Proposition 1 (Skewness-dependent pricing of skewness)**.** *If and only if $a > 0$, there exists $\bar{S} > 0$ such that $\frac{\partial P_X}{\partial S_X} > (=, <) 0$ for $S_X < (=, >) \bar{S}$.*

Proposition 1 states that the price of a negatively or not too positively skewed asset increases in its skewness. This is in line with the intuition that probability weighting results in overweighting rare and extreme events, both bad ones (as come with left-skewed risks) and good ones (as come with right-skewed risks). The formal mechanism behind this result is as follows: If asset X is left-skewed, we have shown in Figure 4 that an increase in the asset's skewness increases its distorted mean and, at the same time, decreases its distorted variance and distorted covariance. All these effects are desirable to an investor with mean-variance preferences with probability distortion, making her willing to pay a higher price.

If asset X is right-skewed, a further increase in S_X continues to increase its distorted mean, but increases rather than decreases its distorted variance and covariance. The first effect dominates for S_X below $\bar{S}$ and the second dominates for S_X above $\bar{S}$. In that latter case, increased skewness results in a lower price. The threshold skewness level $\bar{S}$ depends on the model parameters. For example, the investor cares about the increase in distorted variance and covariance more if parameters $\bar{N}_X$ or γ are larger. We illustrate this result in Figure 13 in Appendix 7.11, which further shows that prices may decrease in skewness for skewness levels that are consistent with no-arbitrage of the Π-CAPM.

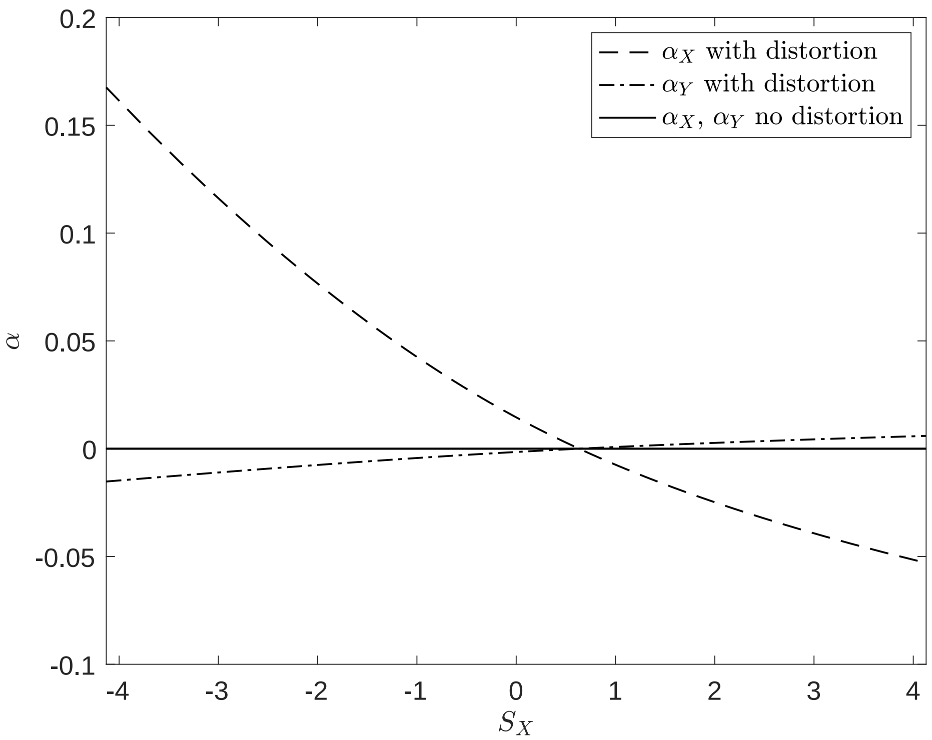
Letting the supply of asset X, $\bar{N}_X$, go to zero, we can compare our result to the pricing of skewness in the model of Barberis and Huang (2008). In their model, the investor overweights the small probability of the large payoff and, therefore, the positively skewed asset has a high price (and low expected return). In our setting, even when the supply of asset X tends to zero, increasing its skewness could lower the price of a right-skewed asset due to the effect of skewness on the distorted covariance, as shown in the right graph of Figure 4. This effect is absent in the model of Barberis and Huang (2008). As the supply of asset X increases, the effect of the distorted variance adds to the covariance effect for right-skewed assets. The result that the pricing of skewness—even under symmetric probability distortion—is not monotonic is new and different from the skewness preference under expected utility.

We now explore the effect of S_X on the risk premium of asset X numerically, using the calibration from Section 2.5. As mentioned before, when conducting comparative statics with respect to skewness, the volatility of the asset's payoff remains unchanged. This is also why Proposition 1 as well as our later price comparative statics results could likewise be stated in terms of asset X's return. Our numerical results, however, are most conveniently illustrated for the assets' alphas defined as

$$\alpha_i = \mathbb{E}R_i - R^f - \beta_i \mathbb{E}(R_m - R^f), i \in \{X, Y\}. \tag{15}$$

One reason is that, as an asset's price changes, the volatility of its return distribution changes slightly.[14] By considering α's, we control for this fact. Moreover, for $a = 0$, the standard CAPM holds so that $\alpha_Y = \alpha_Y = 0$, which offers a single and easy-to-interpret benchmark for both assets.

Figure 5: The effect of S_X on the α's of assets.
This figure illustrates the effect of S_X on the α of each asset. The dashed and dash-dotted line represent α_X and α_Y in case the investor distorts probabilities. The solid line represents the α_X and α_Y in case the investor does not distort probabilities. The scale is in percentages per annum.



[14]Indeed, the graphs for returns that correspond to those in Figures 5, 6, and 10 look qualitatively and quantitatively very similar. Because asset X is small, so is the aforementioned effect of changing return volatility.

In Figure 5, we represent α_X and α_Y as a function of S_X. The solid lines show that, in the absence of probability weighting, α_X and α_Y are equal to zero. The dashed line and dash-dotted line represent α_X and α_Y in the case in which the investor distorts probabilities. In that case, α_X is decreasing in S_X, both when asset X is left-skewed or right-skewed. That is, for the parameters assumed in Figure 5, we are in the case of Proposition 1 in which the price of asset X is decreasing in S_X. The effect of S_X on α_X is economically sizable. As skewness increases from $S_X = -1.5$, to $S_X = 1.5$, α_X decreases from 5% to -2.5% per annum. Figure 5 further shows that the effect of S_X on α_X is diminishing in its level (i.e., α_X is convex in S_X). The reason is that, as explained, a more asymmetric asset X has greater distorted variance and covariance, making it less desirable. For even larger S_X than shown in Figure 5, α_X would increase in S_X, as we show in Proposition 1. The effect of S_X on α_Y is positive and smaller than that on α_X just discussed, which is driven by the fact that the value-weighted sum of α_X and α_Y equals zero. However, since the supply of asset X ($\bar{N}_X$) is small but not infinitesimally small, unlike in Barberis and Huang (2008), the skewness of asset X does impact the equilibrium price of Y.

The result that larger skewness typically decreases expected returns is documented in the empirical asset pricing literature in various markets. Boyer et al. (2010) find that stocks with large idiosyncratic skewness earn low expected returns. Furthermore, Conrad et al. (2013) show that stocks with a positively skewed return distribution, proxied with risk-neutral skewness, earn low expected returns. Boyer and Vorkink (2014) document that also in individual options markets there is a strong negative relationship between total skewness of the option and its expected returns; see also Schneider (2015).[15] We are not aware of empirical work that examines whether the relationship between skewness and returns is non-monotonic for high skewness levels, and we leave this empirical question for further research.[16]

[15]Other empirical papers that discuss the effect of skewness on returns are Kumar (2009), Bali et al. (2011) and Ghysels et al. (2016).

[16]In an experimental setting, Ebert (2015) has observed that skewness preference is less pronounced for (extremely) right-skewed risks.

3.2 Skewness-dependent pricing of volatility

In this section, we explore the effect of the volatility of asset X on its price. From the expressions for the distorted moments in equations (9) to (11) it is clear that a change in the objective volatility of asset X affects its distorted mean and distorted volatility whenever X is skewed. We formalize the effect of volatility of asset X on its price in Proposition 2 below. Statement 1 assumes $\gamma > 0$ sufficiently small, an assumption we occasionally also make later in the paper. We impose the assumption if the effect of risk (or variance) aversion can be opposite to that of probability weighting and dominate it. γ small but not zero ensures that the effect of probability weighting dominates that of risk aversion so that we can learn about the novel force at work within the Π-CAPM.

Proposition 2 (Skewness-dependent pricing of volatility)**.**

1. *If and only if* $a > 0$*, for* $\gamma > 0$ *sufficiently small* $\dfrac{\partial P_X}{\partial \sigma_X} > (=, <) 0$ *for* $S_X > (=, <) 0$.
2. *If and only if* $a > 0$*, there exists* $\bar{S} > 0$ *such that* $\dfrac{\partial^2 P_X}{\partial \sigma_X \partial S_X} > (=, <) 0$ *for* $S_X < (=, >) \bar{S}$.

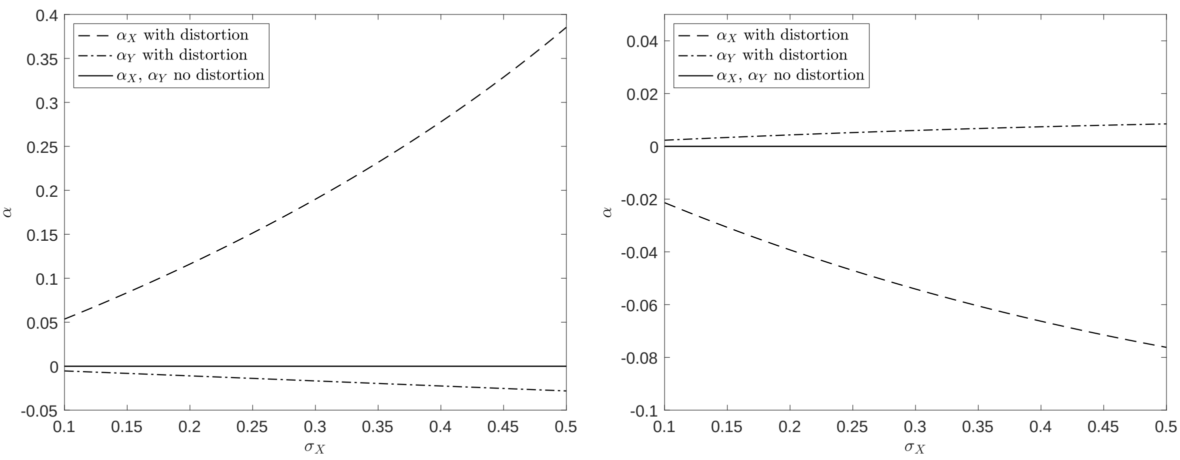
The first statement of Proposition 2 states that the price of the skewed asset increases (decreases) in its volatility if the asset is right-skewed (left-skewed) for risk aversion sufficiently small. Intuitively, under probability weighting, volatility may be desirable or undesirable—depending on the asset's skewness and how strong probability weighting is relative to risk aversion. In Theorem 1, we have shown that prices increase in distorted mean and decrease in distorted variance and distorted covariance. From equation (7) it immediately follows that the first effect dominates if risk aversion is sufficiently small. Therefore, we effectively show that the distorted mean of the skewed asset increases (decreases) in its volatility if the asset is right-skewed (left-skewed): The price of volatility is skewness-dependent.[17]

The second statement of Proposition 2 states that the price change of asset X with respect to volatility increases in S_X for S_X below $\bar{S}$ (in particular, S_X being negative).

[17]Naturally, larger γ makes volatility more undesirable for left-skewed assets and less desirable or even undesirable for right-skewed assets. Statement 1 of Proposition 2 thus also well illustrates the role of the "γ sufficiently small" assumption. Very large values of γ would simply make volatility undesirable, dominating the differential effect that probability weighting brings into the Π-CAPM.

That is, for negatively and not too positively skewed assets, volatility is disliked less the more right-skewed the asset is. For right-skewed assets, the interpretation of the cross-derivative is analogous, but one must keep in mind that increased skewness and volatility may or may not, respectively, increase price in that case (Propositions 1 and 2). Figure 6 illustrates the economically most interesting case graphically.

Figure 6: The effect of volatility of X on the α of the assets.
This figure illustrates a case where α_X is increasing in volatility (left graph) and one where it is not (right graph). The left graph represents the case in which $S_X = -3$, whereas the right graph represents the case in which $S_X = 3$. The dashed and dash-dotted line represent α_X and α_Y in case the investor distorts probabilities. The solid line represents the α_X and α_Y in case the investor does not distorts probabilities. The scale is in percentages per annum.



The dashed lines in Figure 6 illustrate the skewness-dependent pricing of volatility (statement 1 of Proposition 2). The left graph of Figure is for $S_X = -3$ and shows that α_X is increasing in the asset's volatility. As the volatility of asset X increases from 10% to 40%, α_X increases from 5% to 25% (per annum), indicating an economically sizable effect. The weighted sum of α_X and α_Y equals zero; therefore, as α_X increases in the volatility of asset X, α_Y decreases (dash-dotted line). The right graph of Figure 6 is for $S_X = 3$ and shows that, under probability distortion, α_X is decreasing in the volatility of asset X. This case thus constitutes an example in which the price of the right-skewed asset is increasing in its volatility. The result is economically sizable: As the volatility of asset X increases from 10% to 40%, α_X decreases from -2% to -7% per annum. Asset prices reflecting volatility aversion for left-skewed assets and volatility-seeking for right-skewed assets are at odds with the predictions of expected utility with positive risk-aversion, because an investor with these type of preferences would be averse to volatility for left-skewed *and* right-skewed assets.

A prominent asset pricing puzzle is the idiosyncratic volatility puzzle described in Ang et al. (2006), who find a negative relation between idiosyncratic volatility of individual stock returns and subsequent alpha of the stock. This finding is in sharp contrast to classical theorems of finance, which predict that idiosyncratic volatility does not affect its expected return or alpha. The Π-CAPM, however, explains the result for right-skewed assets, whose alpha's may decrease in their volatility. The graphs of Figure 5 thus present a new prediction for the cross-section of stock returns: The alpha for a right-skewed asset is decreasing in its volatility, whereas for a left-skewed asset the alpha is increasing in its volatility.

3.3 Left-skewed *and* right-skewed assets have a positive variance premium

The variance premium of the stock market being positive (i.e., its risk-neutral variance exceeding the objective one) is empirically well-documented (Carr and Wu 2009; Kozhan et al. 2013). While difficult to reconcile with standard asset pricing models, in this section we show that the Π-CAPM predicts a positive variance premium for sufficiently asymmetric individual assets (either sufficiently left-skewed or right-skewed). Particularly the result for right-skewed assets is noteworthy, because, in the absence of probability weighting this variance premium will be negative. Moreover, we find that the size of an asset's variance premium increases in its asymmetry. These predictions, based on the Π-CAPM, are new and testable. In section 4, we report very supportive evidence for them utilizing data from the cross-section of individual stock options.[18]

Theorem 2 presented the risk-neutral distribution for the Π-CAPM and illustrated its dependence on skewness in equation (13). Consequently, the skewness of asset X has pricing implications for derivatives signed on it. Following Bollerslev et al. (2009) and Kozhan et al. (2013), we define the variance premiums as a quadratic contract with the return of asset X as the underlying.[19]

[18]Baele et al. (2019) show that probability weighting predicts a large variance premium of the S&P 500 and verify this prediction using index options. The Π-CAPM allows for multiple assets and makes predictions regarding the variance premium of individual stocks. In particular, we show that (and how) the size of the variance premium of an individual stock varies with its skewness, and we verify this new prediction using the cross-section of individual options.

[19]We consider a squared yearly return contract. The price of such a contract is approximately equal to the price of a variance contract if daily returns are uncorrelated. The prices are exactly equal in case

Definition 1. *The variance premium is defined as follows:* $VP_X = \mathbb{E}^{\mathbb{Q}}\Big((r_X)^2\Big) - \mathbb{E}\Big((r_X)^2\Big) = \big(q_X - p_X\big) \cdot \Big((r_{\bar{x}})^2 - (r_{\underline{x}})^2\Big)$*, where* $(r_{\bar{x}})^2$ *and* $(r_{\underline{x}})^2$ *are the squared (simple) return in the good and bad state of asset* X*, respectively, and* $q_X := q_{\bar{x}\bar{y}} + q_{\bar{x}\underline{y}}$*.*

Proposition 3 (Left-skewed *and* right-skewed assets have a positive variance premium)**.**

1. *If* $a = 0$*,* $\rho_{XY} \geq 0$*, and* $S_X \geq 0$*, then* $VP_X < 0$*.*
2. *If* $a > 0$*, for* $\gamma > 0$ *sufficiently small there exists* $\bar{S} < 0$ *such that* $VP_X > 0 \iff S_X \notin (\bar{S}, 0)$*.*

The main result of Proposition 3, its second statement, makes the assumption of sufficiently small risk aversion discussed before when interpreting Proposition 2 on the price impact of volatility. Before we discuss the implications of Proposition 3, it is insightful to discuss the "small γ" assumption's impact on the Π-CAPM's risk-neutral distribution in general.

The risk-neutral distribution of the Π-CAPM reflects the investor's (standard) mean-variance preference as well as probability weighting. First, risk aversion (i.e., variance aversion) affects risk-neutral probabilities similarly to risk aversion in the expected utility model. Low wealth states receive risk-neutral probabilities that exceed their objective probabilities (as if the investor had decreasing marginal utility of wealth). Analogously, the risk-neutral probabilities of high wealth states are decreasing in risk aversion. Second, probability weighting causes overweighting of extreme, small probability states—be they good or bad—and thus increases the risk-neutral probability of either. Consequently, probability weighting can have opposite or similar effects on the risk-neutral probabilities:

- Risk aversion *increases* the risk-neutral probability of states with low wealth. Probability weighting *increases* the risk-neutral probability of the extreme bad state.
- Risk aversion *decreases* the risk-neutral probability of states with high wealth. Probability weighting *increases* the risk-neutral probability of the extreme good state.

of uncorrelated daily log returns.

Whether—in the case of higher wealth states—the effect of risk aversion or the effect of probability weighting dominates depends on the relative strength of risk aversion and probability distortion. The assumption of sufficiently small risk aversion then ensures that probability weighting dominates so as to make the effects of the novel force of the Π-CAPM visible. In particular, sufficiently small risk aversion ensures an increase in the risk-neutral probabilities of *all* extreme states.

The first statement of Proposition 3 serves to illustrate that, for reasonable parameter values, standard preferences struggle with the prediction of a positive variance risk premium.[20] The intuition is that the squared-return contract pays the large outcome $(r_{\bar{x}})^2$ in states in which the investor has relatively large wealth, and thus these states have a low price of risk. For the variance premium to be positive, the price of risk has to be high in states with relatively high wealth. Consequently, such a prediction is difficult to reconcile with standard preferences.

The second statement of Proposition 3—the main result, on the impact of probability weighting—states that, for sufficiently small risk aversion, the variance premium is positive if asset X is right-skewed ($S_X > 0$) or sufficiently left-skewed ($S_X < \bar{S}$).[21] The fact that, in our equilibrium model, the expected return is usually not zero but positive leads to the slight asymmetry regarding a negative variance premium for mildly left-skewed assets; for details see the proof of Proposition 3 in the appendix. The prediction that left-skewed assets have a positive variance premium holds irrespective of the inclusion of probability weighting. In the case of probability weighting, however, the variance premium is amplified significantly (see also the numerical illustration in Figure 7 below). In contrast, the prediction that right-skewed assets have a positive variance premium

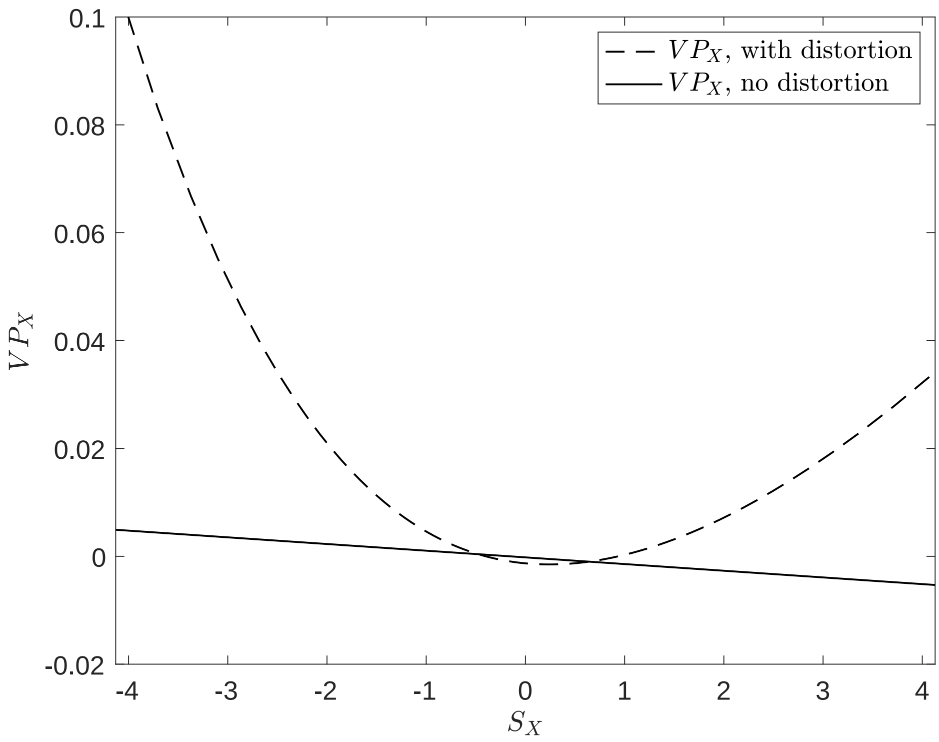
[20]To formally see why, note that, under the assumption of positive correlation, the risk-neutral probability of the state in which asset X pays the high outcome is lower than the objective probability of this state. It follows that the first factor in the calculation of the variance premium, $q_X - p_X$, is negative. Under the assumption that asset X is right-skewed, the second factor in the calculation of the variance premium $(r_{\bar{x}})^2 - (r_{\underline{x}})^2$ is positive. In total, the variance premium is negative.

[21]To get the formal intuition for this result, first note that the variance premium is positive if the two factors $q_X - p_X$ and $r_{\bar{x}}^2 - r_{\underline{x}}^2$ in the calculation of VP_X have the same sign. With risk aversion sufficiently small and probability weighting being symmetric, the first factor is strictly positive if and only if $p_X < 0.5$ because, then, p_X is overweighted and $q_X > p_X$. Therefore, the first factor is strictly positive if and only if X is right-skewed. Regarding the second factor, first note that, for a right-skewed distribution with mean zero, the square of the largest outcome exceeds that of the smallest outcome. Doing the analogous thought experiment with a left-skewed distribution with mean zero, and combining with the previous argument that $q_X > p_X \iff S_X > 0$, yields $VP_X > 0 \iff S_X \neq 0$.

is different from, and unique to, the case of probability weighting. This result is due to the differential impact of probability weighting and risk aversion on the price of risk of each state discussed above. Probability weighting increases the price of risk in small probability states with extreme high and extreme low wealth, whereas risk aversion only increases the price of risk in states with relatively low wealth.

Figure 7 shows that the variance premium predicted by the Π-CAPM is economically sizable for our benchmark calibration and very different from the ordinary CAPM.

Figure 7: The effect of S_X on the variance premium of asset X.
This figure illustrates the effect of S_X on the variance premium of asset X. We plot the variance premium of Definition 1 for the skewed asset X as a function of S_X. The dashed line corresponds to our benchmark calibration with probability distortion and the solid line is in absence of probability distortion.



For right-skewed assets, the variance premium is negative in the absence of probability weighting (solid line), in line with the first statement of Proposition 3. With probability weighting, for our model calibration with risk aversion level well above zero ($\gamma = 1.5$), the asset has to be sufficiently right-skewed for the variance premium to be positive. Then, the effect of probability weighting dominates the effect of risk aversion, and yields a positive variance premium. The effect of probability weighting on the variance premium is economically sizable. For $S_X = -2$, the variance premium equals 0.02, meaning that the investor has to pay a premium of approximately $33\% = 0.02/0.06$ p.a. to hold the squared return contract for asset X.

If asset X is sufficiently left-skewed, the variance premium is positive regardless of the presence of probability weighting. In the absence of probability weighting, the risk-neutral probability of the extreme (bad) state is lower due to risk aversion. With probability weighting, the risk-neutral probability is further increased and, therefore, the variance premium is significantly amplified for left-skewed assets in the Π-CAPM.

3.4 A right-skewed asset has a positive skewness premium

In this section, we analyze the predictions of the Π-CAPM with respect to the skewness premium. An asset's skewness premium is an important determinant of the prices of options signed on the asset (to be discussed in detail at the end of this section). Kozhan et al. (2013) have found that the skewness premium of the market is negative, and strongly related to the variance premium. We show that the Π-CAPM predicts a positive skewness premium for right-skewed assets and a negative skewness premium for left-skewed assets. In particular the former prediction noteworthy, because the standard CAPM predicts a negative skewness premium for right-skewed assets. Unlike the standard CAPM, the Π-CAPM predicts an economically sizable skewness premium for a left-skewed and right-skewed assets alike. In Section 4, we verify these predictions in the cross-section of individual stock options.

In order to study the prediction of the asset's skewness on the skewness premium, we define the skewness premium as the difference between the price and expected payoff of a cubic contract in Definition 2. We formalize the predictions of the Π-CAPM with respect to the skewness premium in Proposition 4.

Definition 2. *The skewness premium is defined as follows:* $SP_X = \mathbb{E}^{\mathbb{Q}}\Big((r_X)^3\Big) - \mathbb{E}\Big((r_X)^3\Big) = \Big(q_X - p_X\Big) \cdot \Big((r_{\bar{x}})^3 - (r_{\underline{x}})^3\Big)$*, where* $(r_{\bar{x}})^3$ *and* $(r_{\underline{x}})^3$ *are the cubic (simple) return in the good and bad state of asset* X*, respectively.*

Proposition 4 (A right-skewed asset has a positive skewness premium)**.**

1. *If* $a = 0$ *and* $\rho_{XY} \geq 0$*, then* $SP_X < 0$.

2. If $a > 0$, for $\gamma > 0$ sufficiently small, then $SP_X < (=, >)0$ for $S_X < (=, >)0$.

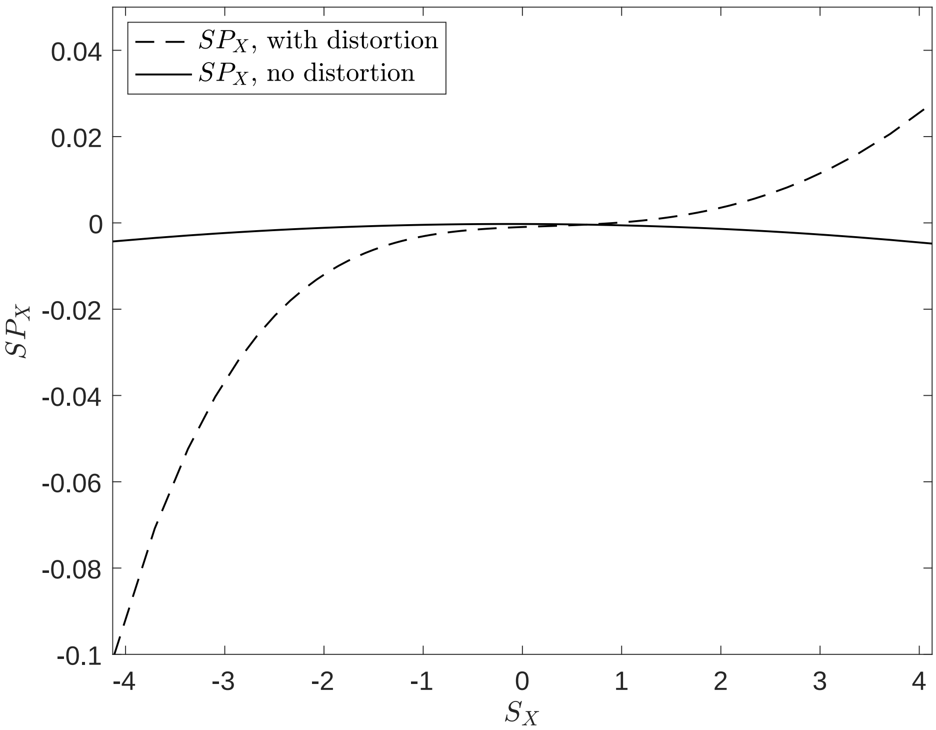
The first statement of Proposition 4 states that, in the absence of probability weighting, the skewness premium is always negative for positively correlated assets X and Y (irrespective of S_X).[22] That is, buying the realized cubic-return is on average profitable. The intuition is that the cubic-return contract pays the large outcome $(r_{\bar{x}})^3$ in states in which the investor has relatively large wealth, and thus these states have a low price of risk. The second statement of Proposition 4 states that, with probability weighting and risk aversion sufficiently small , the skewness premium of asset X is positive (negative) if the asset is right-skewed (left-skewed).[23]

The dashed line in Figure 8 illustrates that, for our benchmark calibration, the skewness premium predicted by the Π-CAPM is sizable.

[22] To get the intuition, first note that the second term in the calculation of the skewness premium, $(r_{\bar{x}})^3 - (r_{\underline{x}})^3$, is always positive, and, therefore, the sign is determined by $q_X - p_X$. If assets X and Y are positively correlated, the risk-neutral probability of the good state of asset Y is always smaller than the objective probability.

[23] Note that, the sign of the skewness premium is determined by the difference in the risk-neutral probability and objective probability of the good state of asset X. If asset X is right-skewed, the risk-neutral probability of the good state is, for sufficiently small risk aversion, larger than the objective probability, and vice versa if asset X is left-skewed.

Figure 8: The effect of S_X on the skewness premium of asset X.
This figure illustrates the effect of S_X on the skewness premium of asset X. We plot the skewness premium of Definition 2 for the skewed asset X as a function of S_X. The dashed line corresponds to our benchmark calibration with probability distortion and the solid line is in absence of probability distortion.



It is increasing in S_X, negative for S_X negative, and positive for S_X positive. In contrast, without probability weighting, the skewness premium is small in absolute size, non-monotone in S_X, and negative also for S_X is large. That is, for a right-skewed underlying, probability weighting changes the skewness premium's sign, its size, and its slope with respect to skewness.

We close this section with a discussion of the implications of our results on the variance and skewness premium for the pricing of put an call options. In particular, the Π-CAPM's predictions regarding the variance and skewness premiums' dependence on the skewness of asset X translate into predictions regarding the prices of puts and calls signed on asset X (and how they depend on its skewness). The reason is that, to earn the variance premium, the investor could buy a portfolio of out-of-the-money (OTM) call options and OTM put options, as shown in Britten-Jones and Neuberger (2000). Similarly, to earn the skewness premium, the investor could buy OTM call options but *sell* OTM put options, as shown in Kozhan et al. (2013). Appendix 7.12 recalls the formal relationships between the variance and skewness premiums and put and call option prices.

With this view of the variance and skewness premiums as option portfolios, first consider the case in which asset X is left-skewed. Π-MV preferences imply a high willingness to pay for the OTM put options, due to overweighting the probability of the stock doing poorly. Therefore, the price of a portfolio with the OTM put options is larger than its expected payoff. Thus, the positive variance premium of a left-skewed asset indicates large put option prices. A similar intuition holds for the skewness premium. The fact that (expensive) OTM put options are sold and (less expensive) OTM call options are bought results in a negative skewness premium.

Second, if the underlying asset X is right-skewed, Π-MV preferences imply a high willingness to pay for OTM call options, due to overweighting the probability of the stock doing very well. Consequently, OTM calls are relatively more expensive than puts. Since both the replicating portfolios of the variance premium and the skewness premium are long in OTM calls, both are positive.

3.5 Positive correlation premium

In this section, we show that the Π-CAPM predicts a positive correlation premium and illustrate using our calibration that it decreases in the size of correlation. The correlation premium is defined as the difference between risk-neutral correlation and actual correlation. First, we show that the covariance between assets is exaggerated if the investor distorts probabilities. This "covariance exaggeration" has pricing implications and leads to a positive correlation premium, in line with existing empirical work, and contrary to the prediction without probability distortion .

We first show formally that probability weighting results in exaggerating the co-movement of assets.

Proposition 5 (Probability weighting exaggerates covariance)**.** *If and only if* $a > 0$: $\mathrm{cov}^{\Pi}(X, Y) > \mathrm{cov}(X, Y)$.

Proposition 5 states that the covariance used for pricing assets is larger than the objective covariance. A consequence of Proposition 5 is that probability weighting has pricing implications even when it is symmetric (as assumed throughout) and even when none of the assets is skewed ($S_X = S_Y = 0$). That is, probability weighting has asset pricing implications beyond the (surely important) re-evaluation of skewness. Next, we define the correlation premium and formalize the predictions of the Π-CAPM with respect to the correlation premium in Proposition 6.

Definition 3. *The correlation premium is defined as follows:* $CP = \text{corr}^{\mathbb{Q}}\big(X, Y\big) - \text{corr}\big(X, Y\big) = \rho^{\mathbb{Q}}_{XY} - \rho_{XY}$. *The risk-neutral probabilities follow from equation* (13).

Proposition 6 (Positive correlation premium)**.**

1. *If* $a = 0$*, then* CP *can be negative.*
2. *If* $a > 0$ *and* $\gamma > 0$ *sufficiently small, then* $CP > 0$*.*

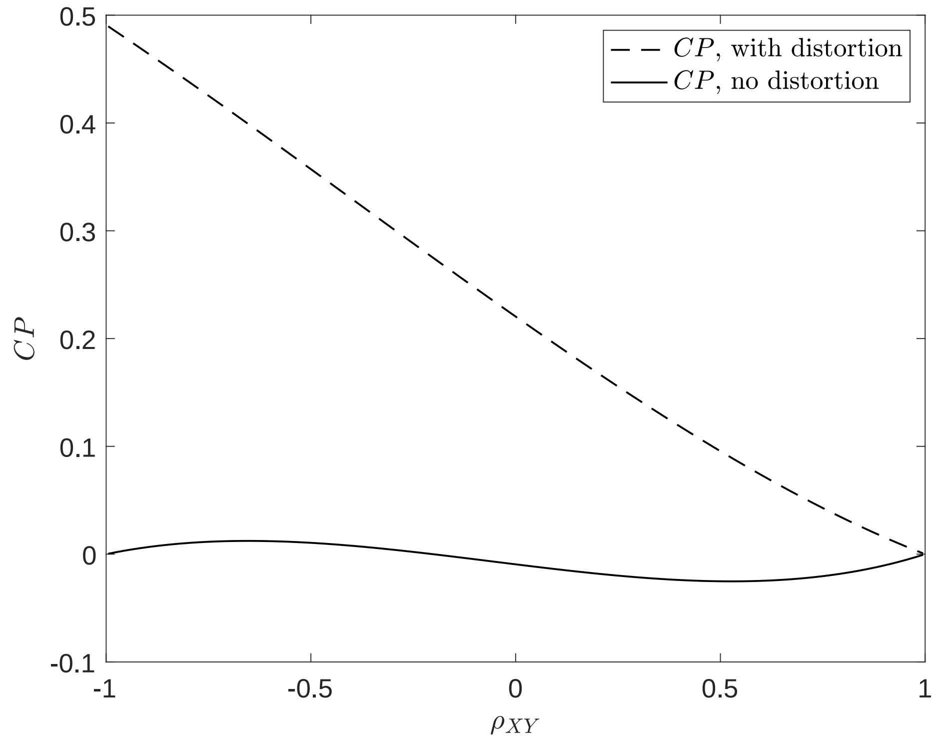
The first statement of Proposition 3 shows that, without probability weighting, the correlation premium can be negative. Figure 9 illustrates this negativity for our benchmark calibration.[24] These results are at odds with empirical studies discussed in detail below, who find a positive and economically sizable correlation premium for positively correlated individual stocks.

The second statement of Proposition 6 states that, with probability weighting (and risk aversion γ sufficiently small), the correlation premium is positive. Intuitively, an investor with probability weighting exaggerates the objective correlation (Proposition 5), which yields a risk-neutral correlation that is larger than the (actual) objective correlation. Specifically, in equilibrium, the investor holds the supply of both assets and, therefore, the extreme states of the economy are the states where both assets pay the good and bad payoff, respectively. Probability weighting makes the investor overweight small probabilities of extreme states. That is, she overweights the probabilities in which the assets move in the same direction—she exaggerates the objective correlation. As explained before, risk aversion sufficiently small isolates the effect of probability weighting on the risk-neutral distribution.

[24]Theoretically, Buraschi et al. (2014) and Buss et al. (2017) show that models with investor disagreement or long-run risks can explain empirical features of the correlation premium. In the absence of investor disagreement or in the case of standard CRRA preferences, the models predict a correlation premium close or equal to zero.

Figure 9 illustrates the results of Proposition 9 for our benchmark calibration and $p_X = p_Y = 0.50$. Notably, contrary to the case without probability weighting, with probability weighting, the correlation premium is economically sizable and positive.

Figure 9: The effect of correlation on the correlation premium.
This figure illustrates the effect of the correlation between the assets and the correlation premium. We plot the correlation premium of Definition 3 as a function of the correlation between the assets. The dashed line corresponds to our benchmark calibration with probability distortion and the solid line is in absence of probability distortion.



For a positive correlation between the assets of 0.5, the risk-neutral correlation equals 0.6, which results in a positive correlation premium of 0.1. In the absence of probability weighting the correlation premium is approximately equal to zero, and even mildly negative in case of positively correlated assets. Therefore, the Π-CAPM and the standard CAPM have very different predictions in case of positively correlated assets. In sum, the Π-CAPM predicts that the correlation premium is sizable, decreases in objective correlation, and approaches zero as objective correlation approaches one.

The Π-CAPM predicting a significant and positive correlation premium (unless objective correlation is close to one) is in line with empirical evidence. In particular, Driessen et al. (2009) and Buss et al. (2016) show that an option strategy that sells correlation risk on the S&P 100 and S&P 500, respectively, earns an economically sizable premium. In contrast to what is documented in the data, the standard CAPM predicts a negative correlation premium for positively correlated assets. Further, Buss et al. (2016) show that the correlation premium of an industry is negatively correlated with the average correlation of the stocks within this industry. Mueller et al. (2017) document a correlation premium in currency markets and, in line with Π-CAPM, find a negative cross-sectional association between average currency correlations and the average correlation premium.

3.6 Relative skewness matters for asset prices

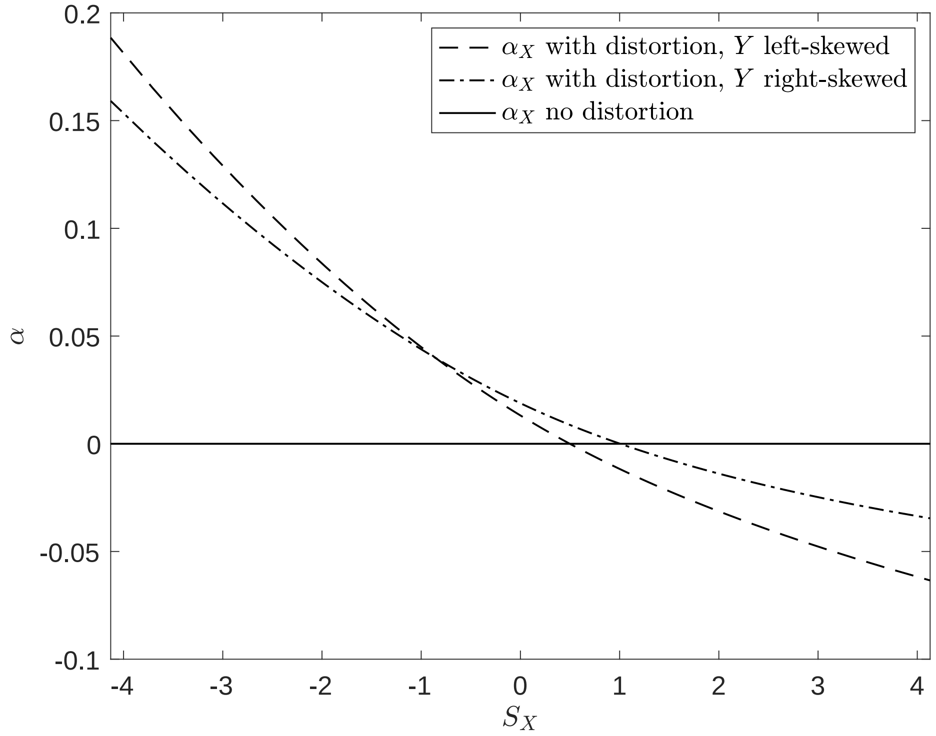
In this section, we investigate the effect of relative skewness on the price of the skewed asset. Here, relative skewness refers to how skewed asset X is relative to the other asset Y (and vice versa). In particular, we are interested in whether asset X is more or less expensive depending on the skewness of asset Y. Therefore, in this section we relax the assumption $p_Y = 0.50$ and allow Y to be either left- or right-skewed. Proposition 7 implies that the Π-CAPM predicts that a right-skewed asset X is more expensive if the other asset Y is moderately left-skewed as opposed to moderately right-skewed:

Proposition 7 (Relative skewness matters for prices)**.** *If and only if $a > 0$, there exists $\bar{S}_1 < 0$ and $\bar{S}_2 > 0$ such that $\dfrac{\partial^2 P_X}{\partial S_X \partial S_Y} < 0$ for $S_Y \in (\bar{S}_1, \bar{S}_2)$.*

Proposition 7 states that, for asset Y sufficiently symmetric, the derivative of the price of asset X with respect to S_X is lower if asset Y is right-skewed rather than left-skewed. Intuitively, a more positively skewed asset is more desirable when the market (excluding the skewed asset) is slightly left-skewed. At the same time, a more negatively skewed asset is more undesirable when the market (excluding the skewed asset) is slightly left-skewed.

To assess the economic significance of these effects, we again resort to our benchmark calibration. We vary the skewness of asset Y and assess its effect on α_X. Figure 10 shows the result of two cases, one in which Y is left-skewed and one in which Y is right-skewed.

Figure 10: The effect of relative skewness on α_X.
This figure shows α_X as a function of S_X in the case in which Y is left-skewed (dashed line) and we plot α_X as a function of S_X in the case in which Y is right-skewed (dash-dotted line). We use the following parameters the figure. $S_Y = 1.15$ corresponds to asset Y right-skewed and $S_Y = -1.15$ corresponds to asset Y left-skewed. The scale is in percentages per annum.



First, Figure 10 shows that α_X decreases in S_X both when asset Y is moderately left-skewed and when it is moderately right-skewed. The magnitude of the effect of S_X on α_X is smaller when Y is right-skewed (i.e., the dash-dotted line declines less quickly). In line with Proposition 7, the investor cares less about additional skewness of asset X when asset Y is already right-skewed. Second, the difference between the two curves in Figure 10 is relatively small compared to how strongly they are decreasing. That is, the effect of relative skewness on the price of asset X is smaller than the effect of its own skewness on price (see also in Section 3.1). For asset X having skewness $S_X = 2$, α_X increases by 2% per annum if the skewness of asset Y increases from $S_Y = -1.15$ to $S_Y = 1.15$. Moreover, if S_X is lower than approximately -1, the difference in α_X becomes negative. Third, the fact that the curves cross as S_X increases can be interpreted as follows. Preference toward the skewness of asset X—be it the aversion to its returns being very left-skewed or the preference for its returns being very right-skewed—is more pronounced when the other asset is left-skewed: The investor accepts larger negative (lower positive) α for a right-skewed (left-skewed) X when Y is left-skewed.

The result of a stronger skewness preference when the other asset is left-skewed has interesting time-series implications. Expected returns for right-skewed assets should be lower in times when the market is more left-skewed. While we are not aware of any evidence for this new asset pricing prediction, the effect of relative skewness is arguably in line with the results of Gao and Lin (2015), who show that trading volume in Taiwanese lottery-like stocks and the lottery jackpot are negatively correlated.[25] Furthermore, in a laboratory experiment Dertwinkel-Kalt and Köster (forthcoming) found that how skewed two lotteries are relative to one another matters for choice.

4 Empirical analysis of the cross-section of variance and skewness premiums

In this section, we take the Π-CAPM's predictions from Sections 3.3 and 3.4—those regarding the variance and skewness premium—to the data. Specifically, Proposition 3 states that the variance premium is positive for left- and right-skewed assets alike, and Figure 11 shows that it increases in the asymmetry (i.e., in the absolute of skewness) of the underlying asset's distribution. Proposition 4 states that the skewness premium is negative for a left-skewed asset and positive for a right-skewed asset, and Figure 12 shows that its absolute value increases in the asymmetry of the underlying asset's distribution. We find economically and statistically significant support for all of these predictions. A further prediction from Figures 11 and 12 is that the effects are economically larger for left-skewed than for right-skewed assets, which we confirm for the variance premium but not for the skewness premium (a null result obtains). Particularly the results on right-skewed assets constitute strong support for the Π-CAPM, because, without probability weighting, variance and skewness premiums are predicted to be negative. At the same time, while a model without probability gets the sign of the premiums for left-skewed assets right, their size is difficult to reconcile without probability weighting.

Besides supporting the Π-CAPM, we believe that our results on the variance

[25]The fact that the trading volume in lottery-like (right-skewed) stocks decreases when the jackpot of the lottery increases (becomes more right-skewed) can be explained by the fact that the preference of investors for right-skewed stocks decreases when other assets are more right-skewed.

and skewness premiums are interesting in their own right. Documenting that (and how) the variance and skewness premiums of individual stock options crucially depend on the skewness of the underlying stock contributes to the recent literature on variance and skewness risk premiums, as discussed in the introduction. In the remainder of this section, we explain and validate our empirical methodology, and afterward we present our empirical results.

4.1 Empirical methodology

Our approach to estimating variance and skewness premiums is close to that of Kozhan et al. (2013). Because applying the methodology of Kozhan et al. (2013) to calculate the variance and skewness risk premiums of individual companies (rather than of the S&P 500, as they do) raises the potential concern of data availability. Carr and Wu (2009) show that for the 35 companies with the most option quotes, the number of available strikes is lower than for the S&P 500.[26] Moreover, the number of available strikes varies significantly across companies and also over time (with an increasing trend). In order to alleviate these concerns, we infer option prices from the volatility surface of OptionMetrics, which covers a fixed number of put and call options that are interpolated from the raw option pricing data. Below we discuss the data and methodology in more detail and show that, when (for the sake of validating our methodology) estimating the variance and skewness premium of the S&P 500, we match the results of Kozhan et al. (2013) closely. Afterward, we apply it to estimate the variance and skewness premiums of individual stocks.

4.2 Data preparation

In this section, we discuss the data preparation in more detail. We use daily option pricing data for individual companies that are included in the S&P 500 during the period 01-1996 to 12-2017. We focus on stocks included in the S&P 500 as these are

[26] In their Table 1 (sample period 1996-2003), Carr and Wu (2009) show that even for the 35 individual stocks with the most quotes available, the average number of available strikes equals approximately 10, whereas for the S&P 500 it equals 26.

large companies with relatively liquid options. The data come from OptionMetrics and we use the daily volatility surface for options with one-month maturity. The volatility surface contains data on put and call options with delta in the range of $[-0.80, -0.20]$ and $[0.20, 0.80]$, respectively. From the definitions of the variance and skewness swap (see the next section and Appendix 4), the fixed rate of each swap depends on OTM option prices with deltas close to zero. For this reason, we inter- and extrapolate the volatility surface to a grid up to option deltas of $+/-0.01$, using cubic splines analogous to Chang et al. (2013). Further, from equation (40) in Appendix 4, the floating rate of the skewness swap depends on daily price changes in the entropy contract. The value of the entropy contract is computed from option prices, and we need option prices with maturity between one month and one day in order to calculate its daily changes. To do so, we assume that the term structure of implied volatility is flat for options with maturity between one month and one day. Because the volatility surfaces from OptionMetrics describe the implied volatilities of European options, in order to calculate option prices, one needs the underlying's dividend yield and the risk-free rate. We assume that the dividend yield of each stock is equal to the average dividend yield over the period for which we observe the volatility surface. As the risk-free rate we use the zero-coupon yield curve from OptionMetrics and interpolate it to the appropriate maturity.

4.3 Validation of methodology: Replicating Kozhan et al. (2013)

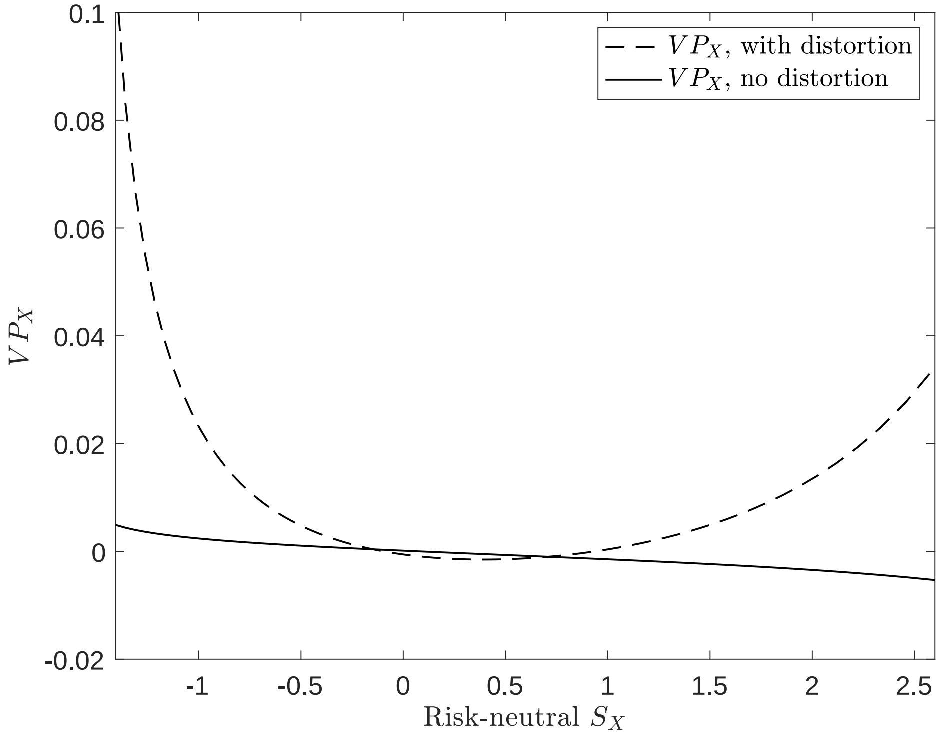
In this section, we show that applying our methodology to estimate variance and skewness premiums of the S&P 500, replicates the findings of from the seminal contribution of Kozhan et al. (2013) closely. This replication suggests that using the volatility surface instead of the raw option is acceptable for the estimation of skewness and variance risk premiums. The detailed results are relegated to Table 4 of Appendix 7.13. As a side result, we can easily extend the sample period of Kozhan et al. (2013) by six years, and document the robustness of their findings for the recent past. Further, in Table 4 we show that our methodology slightly underestimates the risk-neutral skewness of the S&P 500. This underestimation is likely driven by the fact that our extrapolation is not able to fully capture the steep volatility smile of S&P 500 index options. However, Bakshi et al. (2003) show that the volatility smile for individual stock options is less pronounced than for index options, which alleviates this underestimation. Overall, despite the difference in methodology, we match the results of Kozhan et al. (2013) closely. An advantage of our method is that it is relatively straightforward to implement—and it aids itself to the analysis of variance and skewness premiums of individual options.

4.4 Empirical results for the variance premium

We first formulate exact testable hypotheses regarding the dependence of an individual stock's variance premium on its skewness, in line with the predictions of the Π-CAPM. Afterward, we verify that the variance premium is positive for left-skewed stocks (Empirical result VP1), positive for right-skewed stocks (Empirical result VP2), and larger for left-skewed stocks (Empirical result VP3). We further show that all these results hold relatively stronger for more asymmetric stocks (e.g., the variance premium of left-skewed stocks increases in their left-skewness.) While all analysis is based on instruments with one month to maturity, unless noted otherwise we report annualized estimates, as they allow for better comparisons with our model predictions.

Following Conrad et al. (2013), we proxy the skewness of an asset's return distribution with its risk-neutral (standardized) skewness. Figure 11 illustrates the dependence of the variance premium on its risk-neutral skewness, as predicted by the Π-CAPM.

Figure 11: The effect of risk-neutral skewness on the variance premium in the Π-CAPM
This figure illustrates the dependence of the (annual) variance premium VP_X of asset X on its risk-neutral skewness, as predicted by the Π-CAPM. The dashed line corresponds to our benchmark calibration with probability distortion and the solid line is in absence of probability distortion.



Note that the curves in Figure 7 (which was for objective skewness) and Figures 11 (for risk-neutral skewness) are very similar. In either case the figure clearly shows that the variance premium is positive for (sufficiently) left- and right skewed assets alike, and that it increases in the asymmetry of the asset's distribution. Quantitatively, the results are more pronounced for negative skewness.

Following Kozhan et al. (2013), we measure risk neutral skewness as implied by option prices, as follows:

$$S_t^{(i)} = \frac{s_t^{(i)}}{\left(vs_t^{(i)}\right)^{\frac{3}{2}}}, \tag{16}$$

where $s_t^{(i)}$ and $vs_t^{(i)}$ correspond to the fixed rate of a skewness and variance swap for stock i at time t that expires in one month respectively, and are defined in equations (41) and (39) of Appendix 7.12. Because the options we use all have a maturity of one month, we suppressed the maturity in the notation. The variance premium of stock i

that is realized at time T(in one month) is measured as $VP_{t,T}^{(i)} := vs_t^{(i)} - rv_{t,T}^{(i)}$, where we calculate the variance swap rate $vs_t^{(i)}$ and the realized variance $rv_{t,T}^{(i)}$ using equations (39) and (38) in Appendix 7.12, respectively. Note, we include T in the notation to indicate that the realized variance premium is observed at maturity of the variance swap (because it depends on the realized variance of the underlying). The realized variance premium of a given stock can be interpreted as the payoff from selling variance swaps (receiving the fixed variance swap rate).

We obtain the following results. Across all time periods and stocks, the variance premium for stocks with a risk-neutral skewness of -0.5 or less equals on average 0.053 ($t = 8.74$), and for stocks with a risk-neutral skewness of 0.5 or more this value is 0.036 ($t = 3.54$).[27] The former is larger than the latter with statistical significance ($t = 2.72$). To obtain a stricter test of the Π-CAPM's prediction regarding the variance premium's dependence on the underlying stock's skewness, we estimate the following cross-sectional regression using Fama-Macbeth:

$$VP_{t,T}^{(i)} = \beta_0 + \beta_1 \cdot S_t^{(i)} \times \mathbb{1}_{S_t^{(i)}<0} + \beta_2 \cdot S_t^{(i)} \times \mathbb{1}_{S_t^{(i)}>0} + \epsilon_t, \tag{17}$$

Our (alternative) hypotheses are that: $\beta_1 < 0$, $\beta_2 > 0$, and $|\beta_1| > |\beta_2|$. The results are shown in Table 2.

Table 2: Cross-sectional regression of variance premium.
This table shows the the Fama-Macbeth estimates of equation (17). The dependent variable is the variance premium measured as $vs_t^{(i)} - rv_{t,T}^{(i)}$. t-statistics are shown in parentheses and are computed using Newey-West standard errors with the number of lags equal to 210.

Coefficient	$VP_{t,T}^{(i)}$
β_0	-0.008
(t-stat)	(-0.54)
β_1	-0.038
(t-stat)	(-4.83)
β_2	0.019
(t-stat)	(5.09)

The results in Table 2 confirm the Π-CAPM's prediction that the variance pre-

[27]The conditional averages coincide with the slope estimates from a Fama-Macbeth cross-sectional regression of the variance premium on the indicator variables $\mathbb{1}_{S_t^{(i)}<-0.5}$ and $\mathbb{1}_{S_t^{(i)}>+0.5}$. The standard errors of the reported t-tests are corrected for cross-sectional correlation, and calculated using Newey-West with 210 lags to account for serial auto-correlation.

mium of left-skewed (right-skewed) stocks increases in left-skewness (right-skewness), consistent with the general U-shaped relation shown in Figure 11. Also, the t-statistic for the null hypothesis $-\beta_1 \leq \beta_2$ equals 3.80, which means that the effect is larger for left-skewed than for right-skewed assets (as predicted by the Π-CAPM).

The economic magnitudes of the coefficients β_1 and β_2 in Table 2 are also significant. For example, if the underlying stock has an expected variance of 0.04 and risk-neutral skewness of -1, then the estimate of -0.038 means that the variance swap rate equals 0.078. Consequently, if for one year an investor hedges variance monthly, the premium equals $(0.040/0.078 - 1)/12 = 4.1\%$ per month.[28]

Empirical results:

VP1 The variance premium is positive for left-skewed stocks.

VP2 The variance premium is positive for right-skewed stocks.

VP3 The variance premium of left-skewed stocks is larger than that of right-skewed stocks.

Moreover, VP1 to VP3 are relatively stronger the more left-skewed resp. right-skewed the considered stocks are.

4.5 Empirical results for the skewness premium

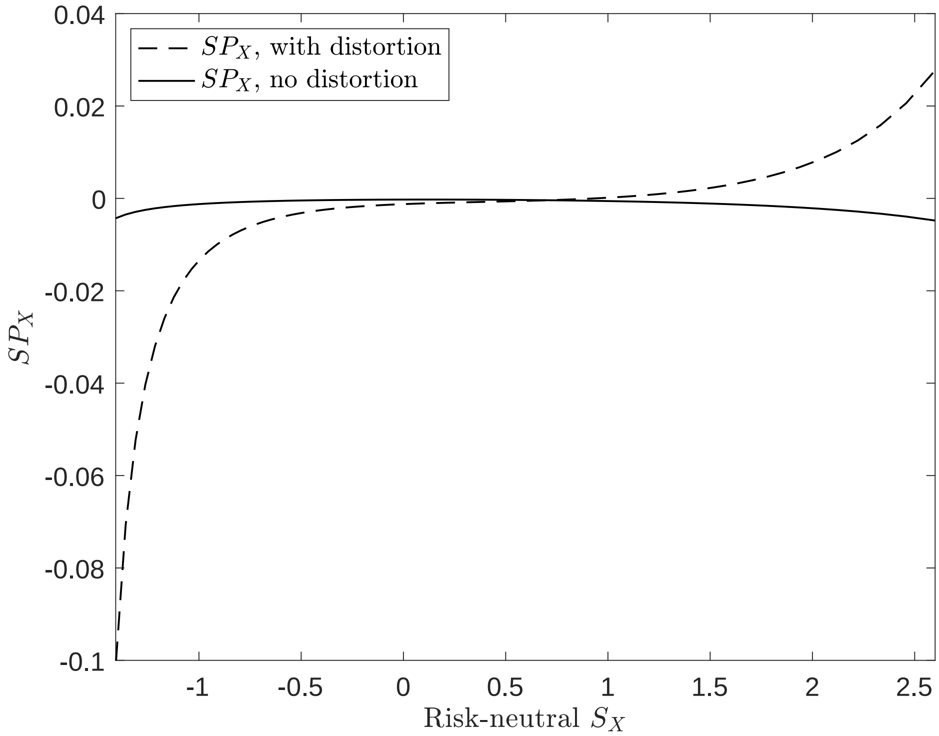
Proposition 4 states that an individual stock's skewness premium is negative if the stock is left-skewed and positive if it is right-skewed. We proceed analogously to the analysis of the variance premium and first formulate exact testable hypotheses, in line with the predictions of Π-CAPM. Afterward, we verify that the skewness premium is negative for left-skewed and positive for right-skewed stocks (Empirical results SP1 and SP2, respectively). A null result obtains regarding a third prediction, being that the

[28]The magnitudes of the model are similar to the data as we consider a squared yearly return contract in the model and the price of such a contract is approximately equal to the price of a variance contract based on daily squared returns if daily returns are uncorrelated. The prices are exactly equal in case of log returns.

absolute of the skewness premium is larger for left-skewed stocks. As with the variance premium, the results hold relative stronger for more asymmetric (left-skewed or right-skewed) stocks.

Similar to the analysis for the variance premium, we first illustrate the dependence of the skewness premium on the underlying asset's risk-neutral skewness; see Figure 12.

Figure 12: The effect of risk-neutral skewness on the skewness premium of asset X.
This figure illustrates the dependence of the (annual) variance premium SP_X of asset X on its risk-neutral skewness, as predicted by the Π-CAPM. The dashed line corresponds to our benchmark calibration with probability distortion and the solid line is in absence of probability distortion.



As was the case for objective skewness (recall Figure 8), the skewness premium increases in risk-neutral skewness when the investor distorts probabilities (dashed line). The skewness premium decreases faster in negative risk-neutral skewness than it increases in positive risk-neutral skewness.

The skewness premium of stock i that is realized at time T is measured as $SP^{(i)}_{t,T} := s^{(i)}_t - rs^{(i)}_{t,T}$, where the realized skewness premium is calculated using equations (40) and (41) in Appendix (7.12). Then, conceptually following the analysis for the variance premium, for stocks with a risk-neutral skewness of -0.5 or less we find an average skewness premium of -0.263×100^{-1} ($t = -11.90$). For stocks with a risk-neutral skewness of 0.5 or more, the average is 0.366×100^{-1} ($t = 3.54$). The absolute value of the former is smaller than the latter, but without statistical significance ($t = -1.46$). We run the regression from equation (17), except for replacing the realized variance premium with the realized skewness premium. Our (alternative) hypotheses are $\beta_1 > 0$, $\beta_2 > 0$, and $\beta_1 > \beta_2$. The results of this cross-sectional regression, which is estimated using Fama-Macbeth, are shown in Table 3.

Table 3: Cross-sectional regression skewness premium.
This table shows (non-annualized) results of the Fama-Macbeth estimates of equation (17). The dependent variable is the skewness premium computed as $s_t^{(i)} - rs_{t,T}^{(i)}$. t-statistics are shown in parentheses and are computed using Newey-West standard errors with the number of lags equal to 210.

Coefficient	$SP_{t,T}^{(i)}$
$\beta_0 \times 100^{-1}$	-0.016
(t-stat)	(-0.29)
$\beta_1 \times 100^{-1}$	0.166
(t-stat)	(6.93)
$\beta_2 \times 100^{-1}$	0.169
(t-stat)	(5.95)

In line with our alternative hypotheses, we observe with statistical significance that β_1 and β_2 are positive. The t-statistic for the null hypothesis $\beta_1 \leq \beta_2$ equals -0.07 and, therefore, we fail to reject that the effect is larger for right-skewed assets than for left-skewed assets.

As regards the assessment of the economic significance of our results on the skewness premium, note that we defined the skewness premium as the difference between the price and expected payoff of a yearly cubic contract. Because it is not readily possible to scale a monthly skewness swap based on daily returns to a yearly skewness swap, we cannot directly compare the cubic yearly contract to the skewness swaps in the data. Hence, the magnitudes of the results in this section cannot be readily compared to those in Figure 12. However, from Table 3 we can infer a large premium for OTM call (put) options on right-skewed (left-skewed) stocks. In combination with our results on the variance premium, this observation indicates that the variance premium for right-skewed (left-skewed) stocks is driven by the expensiveness of OTM call (put) options.

Empirical results:

SP1 The skewness premium is negative for left-skewed stocks.

SP2 The skewness premium is positive for right-skewed stocks.

Moreover, SP1 to SP2 are relatively stronger the more left-skewed resp. right-skewed the considered stocks are.

4.6 Summary of the empirical analysis

In this section we presented some empirical support for the Π-CAPM. We confirmed its predictions that the variance premium is positive for left-skewed and right-skewed assets alike, and that it is larger for left-skewed than for right-skewed assets. We further confirmed the Π-CAPM's predictions that the skewness premium is positive (negative) for positively (negatively) skewed assets. All these results hold relatively stronger for more asymmetric stocks (e.g., the variance premium of left-skewed stocks increases in their left-skewness.)

Especially the results of positive variance and skewness premiums for right-skewed assets constitute strong evidence in favor of the Π-CAPM. Underlying these predictions are preferences that imply a large willingness to pay for securities that pay in large wealth states. This is not the case for standard preferences, for which securities paying in states with large wealth should have low prices.

5 Conclusion

This paper has proposed and analyzed the Π-CAPM—a capital asset pricing model in which objective probabilities are replaced with a collection of decision weights, denoted by Π, as motivated by the behavioral economics literature on probability weighting. The Π-CAPM nests the classical Lintner-Sharpe-CAPM and extends it by one parameter that captures probability weighting. The tractability of the Π-CAPM allows for a number of predictions—some known and some entirely new—regarding the pricing of skewed and correlated assets. In the Π-CAPM, symmetric probability weighting has asymmetric pricing implications, objective correlation is exaggerated, and volatility and skewness affect asset prices in specific ways. While the price of a left-skewed asset increases in skewness and decreases in variance of the asset, the price of a sufficiently right-skewed asset may decrease in skewness and increase in variance of the asset.

One novel prediction of the Π-CAPM is that variance and skewness premiums depend in specific ways on the underlying asset's own skewness. In the empirical part of the paper, we take this prediction to the test and find strong support for it within the cross-section of individual US stock options. We find that the variance premium of individual stocks is positive and increasing in the stock's asymmetry (the absolute of skewness). The skewness premium increases in the skewness of the underlying stock, is negative for left-skewed stocks, and is positive for right-skewed stocks. While the skewness dependence of variance and skewness premiums that we document in the data is difficult to explain with standard preferences (both qualitatively and quantitatively), it is consistent with Π-CAPM.

6 Bibliography

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7 Appendix

In the following subsections we conveniently use the following notation for the decision weights. Similar to joint probability density of the two assets, we derive the marginal components of each asset for the decision weights in the following way:

$$
\begin{aligned}
\pi_X &:= \pi_{\bar{x}\bar{y}} + \pi_{\bar{x}\underline{y}}, \\
\pi_Y &:= \pi_{\bar{x}\bar{y}} + \pi_{\underline{x}\bar{y}}, \qquad (18) \\
r^{\Pi}_{XY} &:= \pi_{\bar{x}\bar{y}} - \pi_X \pi_Y.
\end{aligned}
$$

These decision weights are used in the next section where we derive the equilibrium of the model.

7.1 Proof of Theorem 1

As explained in the main text preceding Theorem 1, the ranking of wealth states is not affected by changes in demand and the value function is differentiable in N_X and N_Y. In equilibrium, demand for either asset must satisfy the respective first-order condition. From equation (4):

$$
\begin{aligned}
\frac{\partial U(W_1)}{\partial N_X} &= \mathbb{E}^{\Pi} X - P_X R^f - \gamma N_X \mathrm{var}^{\Pi}\big(X\big) - \gamma N_Y \mathrm{cov}^{\Pi}\big(X, Y\big) = 0, \text{ and} \\
\frac{\partial U(W_1)}{\partial N_Y} &= \mathbb{E}^{\Pi} Y - P_Y R^f - \gamma N_Y \mathrm{var}^{\Pi}\big(Y\big) - \gamma N_X \mathrm{cov}^{\Pi}\big(X, Y\big) = 0.
\end{aligned}
$$

The second-order condition yields the following:

$$
\begin{aligned}
&\frac{\partial^2 U(W_1)}{\partial N_X^2} \cdot \frac{\partial^2 U(W_1)}{\partial N_Y^2} - \left(\frac{\partial^2 U(W_1)}{\partial N_X \partial N_Y}\right)^2 \geq 0, \\
\iff \quad & \gamma^2 \mathrm{var}^{\Pi}\big(X\big) \mathrm{var}^{\Pi}\big(Y\big) - \gamma^2 \mathrm{cov}^{\Pi}\big(X, Y\big)^2 \geq 0, \\
\iff \quad & -1 \leq \mathrm{corr}^{\Pi}\big(X, Y\big) \leq 1,
\end{aligned}
$$

and indicates that the optimal demand functions obtained from the first-order conditions, indeed, maximize the utility.

Market clearing, $N_X = \bar{N}_X$ and $N_Y = \bar{N}_Y$, yields the pricing equations in terms of distorted moments:

$$P_X R^f = \mathbb{E}^{\Pi} X - \gamma \bar{N}_X \mathrm{var}^{\Pi}\big(X\big) - \gamma \bar{N}_Y \mathrm{cov}^{\Pi}\big(X, Y\big), \text{ and} \tag{19}$$

$$P_Y R^f = \mathbb{E}^{\Pi} Y - \gamma \bar{N}_Y \mathrm{var}^{\Pi}\big(Y\big) - \gamma \bar{N}_X \mathrm{cov}^{\Pi}\big(X, Y\big). \tag{20}$$

Next, we express the distorted moments in terms of non-distorted moments. For the distorted mean we obtain:

$$\begin{aligned}
\mathbb{E}^{\Pi} X &= \pi_X \bar{x} + (1 - \pi_X) \underline{x} = \pi_X (\bar{x} - \underline{x}) + \underline{x} \\
&= \big(a + (1 - 2a) p_X\big)(\bar{x} - \underline{x}) + \underline{x} = a(1 - 2p_X)(\bar{x} - \underline{x}) + \underbrace{p_X(\bar{x} - \underline{x}) + \underline{x}}_{=\mu_X} \\
&= a S_X \sigma_X + \mu_X.
\end{aligned}$$

Similarly distorted variance is given by:

$$\begin{aligned}
\mathrm{var}^{\Pi}(X) &= \pi_X (1 - \pi_X)(\bar{x} - \underline{x})^2 \\
&= \big(a + (1 - 2a) p_X\big)\big(1 - a - (1 - 2a) p_X\big)(\bar{x} - \underline{x})^2 \\
&= \big(a(1 - a) + (1 - 4a + 4a^2) p_X - (1 - 4a + 4a^2) p_X^2\big)(\bar{x} - \underline{x})^2 \\
&= a(1 - a)\big(1 - 4p_X + 4p_X^2\big)(\bar{x} - \underline{x})^2 + p_X(1 - p_X)(\bar{x} - \underline{x})^2 \\
&= a(1 - a) S_X^2 \sigma_X^2 + \sigma_X^2.
\end{aligned} \tag{21}$$

Note that the distorted variance is given by

$$\mathrm{cov}^{\Pi}(X, Y) = r_{XY}^{\Pi} (\bar{x} - \underline{x})(\bar{y} - \underline{y}),$$

where

$$\begin{aligned}
r_{XY}^{\Pi} &= \pi_{\bar{x}\bar{y}} - \pi_X \pi_Y \\
&= \big(a + (1 - 2a) p_{\bar{x}\bar{y}} - \big(a + (1 - 2a) p_X\big)\big)\big(a + (1 - 2a) p_Y\big) \\
&= a(1 - a) + (1 - 2a)\big(p_{\bar{x}\bar{y}} - a p_Y - a p_X - (1 - 2a) p_X p_Y\big) \\
&= a(1 - a) + (1 - 2a) r_{XY} - a(1 - 2a) \underbrace{\big(p_Y(1 - p_X) + p_X(1 - p_Y)\big)}_{=\frac{1}{2} \text{ if } S_Y = 0}
\end{aligned}$$

$$= \frac{1}{2}a + (1-2a)r_{xy}. \tag{22}$$

Therefore,

$$\begin{aligned}
\mathrm{cov}^{\Pi}(X,Y) &= (1-2a)\mathrm{cov}(X,Y) + \frac{1}{2}a(\bar{x}-\underline{x})(\bar{y}-\underline{y}) \\
&= (1-2a)\mathrm{cov}(X,Y) + \frac{1}{2}a\frac{\sigma_X}{\sqrt{p_X(1-p_X)}} \cdot \frac{\sigma_Y}{\sqrt{p_Y(1-p_Y)}} \\
&= (1-2a)\mathrm{cov}(X,Y) + a\frac{\sigma_X\sigma_Y}{\sqrt{p_X(1-p_X)}}.
\end{aligned}$$

The expressions for distorted mean and distorted variance of asset Y are obtained analogously. Inserting all the expressions for the distorted moments into in equations (19) and (20), and exploiting that $S_Y = 0 \iff p_Y = 0.5$, yields the result. ■

7.2 Proof of Theorem 2

We first prove that 2. implies 1. In particular, we assume that $c > 2\sqrt{a(1-a)}$ and $p_X \in (p_1^*, p_2^*)$, where

$$c := \frac{1-\gamma\bar{N}_Y\sigma_Y}{\gamma\bar{N}_X\sigma_X}, \tag{23}$$

$$p_1^* := \frac{(2a-1)(2a-2)+c^2-c\cdot\sqrt{c^2-4a(1-a)}}{2c^2+2(1-2a)^2}, \text{ and} \tag{24}$$

$$p_2^* := \frac{(2a-1)(2a-2)+c^2+c\cdot\sqrt{c^2-4a(1-a)}}{2c^2+2(1-2a)^2}. \tag{25}$$

Let $i \in \{\bar{x}, \underline{x}\}, j \in \{\bar{y}, \underline{y}\}$ and recall that $p_{ij} > 0$ by assumption. Statement 1 follows if (i) $\mathbb{E}^{\mathbb{Q}}X = P_X R^f$, (ii) $\sum_{i,j} q_{ij} = 1$, (iii) $q_{ij} \in [0,1]$, and (iv) $q_{ij} > 0 \iff p_{ij} > 0$. Conditions (ii) and (iii) mean that $\mathbb{Q}$ is a probability measure, (i) means that it is a risk-neutral pricing measure, and (iv) means that it is equivalent to $\mathbb{P}$. By the first fundamental theorem of asset pricing, conditions (i) to (iv) jointly ensure the absence of arbitrage.

We first rewrite the pricing equation (5):

$$\begin{aligned}
P_X R^f &= \mathbb{E}^{\Pi}X - \gamma\bar{N}_X\mathrm{var}^{\Pi}(X) - \gamma\bar{N}_Y\mathrm{cov}^{\Pi}(X,Y) \\
&= \mathbb{E}^{\Pi}X\Big[1 - \gamma\bar{N}_X(X-\mathbb{E}^{\Pi}X) - \gamma\bar{N}_Y(Y-\mathbb{E}^{\Pi}Y)\Big]
\end{aligned}$$

$$= \mathbb{E}^{\Pi} X \Big[1 - \gamma \bar{N}_X (X - \mu_X - a\sigma_X S_X) - \gamma \bar{N}_Y (Y - \mu_Y) \Big]$$

where we made use of equation (9) and $p_Y = 0.5$. Therefore, (i) holds for $\mathbb{Q}$ (and only for $\mathbb{Q}$) as defined through the state probabilities in equation (13).

Because of $\mathbb{E}^{\Pi} \gamma \bar{N}_X (X - \mathbb{E}^{\Pi} X) = 0$ and $\mathbb{E}^{\Pi} \gamma \bar{N}_Y (Y - \mathbb{E}^{\Pi} Y) = 0$, we have that $\sum_{i,j} q_{ij} = \sum_{i,j} \pi_{ij} = 1$. Therefore, (ii) holds.

Next, observe that $q_{ij} > 0$ for all i, j is equivalent to:

$$\begin{aligned}
& 1 - \gamma \bar{N}_X (i - \mu_X - a\sigma_X S_X) - \gamma \bar{N}_Y (j - \mu_Y) > 0 \text{ for all } i, j \\
\iff \quad & 1 - \gamma \bar{N}_X (\bar{x} - \mu_X - a\sigma_X S_X) - \gamma \bar{N}_Y (\bar{y} - \mu_Y) > 0 \\
\iff \quad & \frac{1 - p_X - a(1 - 2p_X)}{\sqrt{p_X(1 - p_X)}} < \frac{1 - \gamma \bar{N}_Y \sigma_Y}{\gamma \bar{N}_X \sigma_X} \equiv c. \qquad (26) \\
\iff \quad & p_X^2 \Big((1 - 2a)^2 + c^2 \Big) - p_X \Big((2a - 1)(2a - 2) + c^2 \Big) + (1 - a)^2 < 0, \qquad (27)
\end{aligned}$$

where the necessity in the last step follows from $c > 2\sqrt{a(1 - a)} > 0$. The assumption that $c > 2\sqrt{a(1 - a)}$ ensures that the the left hand side of inequality (27) has two roots in p_X. They are given by equations (24) and (25). Inequality (27) holds for $p_X \in (p_1^*, p_2^*)$ and thus $q_{ij} > 0$. $q_{ij} < 1$ then follows from (ii), which proves (iii).

(ii) and (iii) together imply (iv).

We now prove that 1 implies 2. By the equivalence of $\mathbb{Q}$ and $\mathbb{P}$ we have $q_{ij} > 0$. Therefore, inequality (27) holds. If $c \leq 2\sqrt{a(1 - a)}$, the left hand side of inequality (27) has no or one root and is (weakly) positive, which would contradict inequality (27). Therefore, $c > 2\sqrt{a(1 - a)}$ and $p_X \in (p_1^*, p_2^*)$. It remains to show that $0 < p_1^* < p_2^* \leq 1$. The first and second inequality are straightforward. Lastly,

$$\begin{aligned}
p_2^* \leq 1 \iff & \frac{(2a - 2)(2a - 1) + c^2 + c\sqrt{c^2 - 4a(1 - a)}}{2(1 - 2a)^2 + 2c^2} \leq 1 \\
\iff & \; 2a(1 - 2a) - c^2 + \sqrt{(c^2 - 2a(1 - a))^2 - (4a(1 - a))^2} \leq 0 \\
\Longleftarrow & \; 2a(1 - 2a) - c^2 + \sqrt{(c^2 - 2a(1 - a))^2} \leq 0. \\
\iff & \; a \geq 0.
\end{aligned}$$

This completes the proof. ■

7.3 Proof of Corollary 1

Noting that $(2a-2)(2a-1) > 2(1-2a)$ for all $a \in [0, 0.5]$ it follows that

$$p_2^* \geq \frac{(2a-2)(2a-1)+c^2}{2(1-2a)^2+2c^2} > 0.5.$$

Because $c > 1 \iff \bar{N}_Y\sigma_Y + \bar{N}_X\sigma_X < \frac{1}{\gamma}$, from inequality (26) it follows that $c > 1 \iff p_1^* < 0.5 \iff 0.5 \in (p_1^*, p_2^*)$. The equivalence statement thus follows from Theorem 2. As regards the statement 1, from the last derivation of the proof of Theorem 2 $p_2^* \leq 1 \iff a = 0$. Because $p_1^* < 0.5$, the l.h.s of inequality (26) is minimized for $a \to 0.5$. As regards the statement 2, note that $c \to \infty \iff \gamma\bar{N}_X\sigma_X \to 0$. The statement then follows from equations (24) and (25). ■

7.4 Proof of Proposition 1

We prove the claim by first showing that $\frac{\partial P_X}{\partial p_X} = 0 \iff a = 0$ or $S_X = \bar{S}$. Afterward, we show that the sign of $\frac{\partial P_X}{\partial S_X}$ changes around the skewness level $\bar{S}$. Because, $\frac{\partial P_X}{\partial p_X} < (=,>) 0 \iff \frac{\partial P_X}{\partial S_X} > (=,<) 0$, we solve for the roots of $\frac{\partial P_X}{\partial S_X}$ as follows:

$$\begin{aligned}
\frac{\partial P_X R^f}{\partial p_X} =& a\sigma_X\frac{\partial S_X}{\partial p_X} - \gamma\bar{N}_X a(1-a)\sigma_X^2 2S_X\frac{\partial S_X}{\partial p_X} \\
&- \gamma\bar{N}_Y a\sigma_Y\sigma_X(1-2p_X)\frac{\partial S_X}{\partial p_X} = 0, \\
\iff & a\sigma_X\frac{\partial S_X}{\partial p_X}\underbrace{\left[1 - 2\gamma\bar{N}_X(1-a)\sigma_X S_X - \gamma\bar{N}_Y\sigma_Y(1-2p_X)\right]}_{=:f(p_X)} = 0, \\
\iff & a = 0 \text{ or } f(p_X) = 0.
\end{aligned}$$

Due to $f(\frac{1}{2}) = 1$ and $\frac{\partial S_X}{\partial p_X} < 0$ it follows that $\frac{\partial P_X}{\partial S_X} > 0$ at $p_X = \frac{1}{2}$. Moreover, because $f(p_X) \to -\infty$ as $p_X \to 0$, it follows, by the mean value theorem, that there exists $\bar{p} : f(\bar{p}) = 0$. Because $\frac{\partial f(p_X)}{\partial p_X} > 0$, f is strictly increasing so that $\bar{p}$ is unique and $\frac{\partial P_X}{\partial S_X} < 0$ for $p_X < \bar{p}$. The claim follows for $\bar{S} := \frac{1-2\bar{p}}{\sqrt{\bar{p}(1-\bar{p})}}$. ■

7.5 Proof of Proposition 2

We prove the first claim by calculating the derivative of the pricing equation of Theorem 1 with respect its volatility:

$$\begin{aligned}\frac{\partial P_X R^f}{\partial \sigma_X} =& aS_X - \gamma \bar{N}_X 2\sigma_X(1 + a(1-a)S_X^2) \\ & - \gamma \bar{N}_Y \sigma_Y \left[(1-2a)\rho_{XY} + \frac{a}{\sqrt{p_X(1-p_X)}}\right].\end{aligned} \tag{28}$$

If $\gamma > 0$ sufficiently small, the sign of the derivative of equation (28) is determined by the sign of S_X. The first claim follows.

We prove the second claim by first showing that $\frac{\partial^2 P_X}{\partial \sigma_X \partial S_X} = 0 \iff a = 0$ or $S_X = \bar{S}$. Afterward, we show that the sign of $\frac{\partial^2 P_X}{\partial \sigma_X \partial S_X}$ changes around the skewness level $\bar{S}$. First, we solve for the roots of $\frac{\partial^2 P_X}{\partial \sigma_X \partial S_X}$, which follows from equation (28):

$$\begin{aligned}& \frac{\partial^2 P_X R^f}{\partial \sigma_X \partial p_X} = 0, \\ \iff & a\frac{\partial S_X}{\partial p_X} \underbrace{\left[1 - 4\gamma \bar{N}_X(1-a)\sigma_X S_X - \gamma \bar{N}_Y \sigma_Y (1-2p_X)\right]}_{=:f(p_X)} = 0, \\ \iff & a = 0 \text{ or } f(p_X) = 0.\end{aligned}$$

Due to $f(\frac{1}{2}) = 1$ and $\frac{\partial S_X}{\partial p_X} < 0$ it follows that $\frac{\partial^2 P_X}{\partial \sigma_X \partial S_X} > 0$ at $p_X = \frac{1}{2}$. Moreover, because $f(p_X) \to -\infty$ as $p_X \to 0$, it follows, by the mean value theorem, that there exists $\bar{p} : f(\bar{p}) = 0$. Because $\frac{\partial f(p_X)}{\partial p_X} > 0$, f is strictly increasing so that $\bar{p}$ is unique and $\frac{\partial^2 P_X}{\partial \sigma_X \partial S_X} < 0$ for $p_X < \bar{p}$. The second claim follows for $\bar{S} := \frac{1-2\bar{p}}{\sqrt{\bar{p}(1-\bar{p})}}$. ■

7.6 Proof of Proposition 3

To prove the first statement, we prove the more general statement:
If $a = 0$, $\rho_{XY} > -\frac{\bar{N}_X \sigma_X}{\bar{N}_Y \sigma_Y}$, then there exists $\bar{S} < 0$ such that $VP_X < (=, >) 0$ for $S_X > (=, <) \bar{S}$.

From Definition 1 we know that $VP_X = \left(q_X - p_X\right) \cdot \left((r_{\bar{x}})^2 - (r_{\underline{x}})^2\right)$. In case

$a = 0$, it follows from equation (13) that the risk-neutral distribution is given by:

$$q_{ij} = p_{ij}\Big[1 - \gamma\bar{N}_X(i - \mu_X) - \gamma\bar{N}_Y(j - \mu_Y)\Big], i \in \{\bar{x}, \underline{x}\}, j \in \{\bar{y}, \underline{y}\}.$$

Therefore, the first term in the calculation of VP_X can be rewritten as:

$$\begin{aligned} q_X - p_X =& p_{\bar{x}\bar{y}}\Big[1 - \gamma\bar{N}_X(\bar{x} - \mu_X) - \gamma\bar{N}_Y(\bar{y} - \mu_Y)\Big] + \\ & p_{\bar{x}\underline{y}}\Big[1 - \gamma\bar{N}_X(\bar{x} - \mu_X) - \gamma\bar{N}_Y(\underline{y} - \mu_Y)\Big] - p_X \\ =& -\gamma\Big[\bar{N}_X p_X(1 - p_X)(\bar{x} - \underline{x}) + \bar{N}_Y r_{XY}(\bar{y} - \underline{y})\Big]. \end{aligned}$$

This term is negative if and only if:

$$\begin{aligned} & r_{XY} > -\frac{\bar{N}_X p_X(1 - p_X)(\bar{x} - \underline{x})}{\bar{N}_Y(\bar{y} - \underline{y})} \\ \Longleftrightarrow & \frac{r_{XY}}{\sqrt{p_X(1 - p_X)}\sqrt{p_Y(1 - p_Y)}} = \rho_{xy} > -\frac{\bar{N}_X\sigma_X}{\bar{N}_Y\sigma_Y} \end{aligned}$$

Next, note that:

$$\begin{aligned} \Big((r_{\bar{x}})^2 - (r_{\underline{x}})^2\Big) = \Big(\frac{\bar{x}}{P_X} - 1\Big)^2 - \Big(\frac{\underline{x}}{P_X} - 1\Big)^2 &= \frac{1}{\big(P_X\big)^2}\Big(\bar{x}^2 - \underline{x}^2\Big) - \frac{2}{P_X}\big(\bar{x} - \underline{x}\big) = 0 \\ \Longleftrightarrow & \big(\bar{x} - \underline{x}\big)\big(\bar{x} + \underline{x}\big) - 2P_X\big(\bar{x} - \underline{x}\big) = 0 \\ \Longleftrightarrow & P_X = \frac{1}{2}(\bar{x} + \underline{x}) = \frac{1}{2}\left(\mu_X + \sigma_X\sqrt{\frac{1 - p_X}{p_X}} + \mu_X - \sigma_X\sqrt{\frac{p_X}{1 - p_X}}\right) \\ \Longleftrightarrow & P_X = \mu_X + \frac{1}{2}\sigma_X S_X, \end{aligned}$$

such that,

$$\Big((r_{\bar{x}})^2 - (r_{\underline{x}})^2\Big) = 0 \Longleftrightarrow \qquad S_X = \frac{\mu_X(1 - R^f) - \gamma\bar{N}_X\sigma_X^2 - \gamma\bar{N}_Y\mathrm{cov}(X, Y)}{\frac{1}{2}\sigma_X R^f} =: \bar{S},$$

which is negative for all $R^f > 1$ if $\rho_{XY} > -\frac{\bar{N}_X\sigma_X}{\bar{N}_Y\sigma_Y}$. Furthermore, it follows that $(r_{\bar{x}})^2 - (r_{\underline{x}})^2 > (=, <) 0$ for $S_X > (=, <)\bar{S}$. The statement follows.

In the following, we prove the second statement. We first determine the sign of the first factor in the calculation of VP_X, which can be rewritten as:

$$
\begin{aligned}
q_X - p_X =& q_{\bar{x}\bar{y}} + q_{\bar{x}\underline{y}} - p_X \\
=& \pi_{\bar{x}\bar{y}}\left(1 - \gamma\Big(\bar{N}_X(\bar{x} - \mathbb{E}^\Pi X) + \bar{N}_Y(\bar{y} - \mathbb{E}^\Pi Y)\Big)\right) + \\
& \pi_{\bar{x}\underline{y}}\left(1 - \gamma\Big(\bar{N}_X(\bar{x} - \mathbb{E}^\Pi X) + \bar{N}_Y(\underline{y} - \mathbb{E}^\Pi Y)\Big)\right) - p_X \\
=& \pi_X - p_X - \gamma \bar{N}_X \pi_X (1 - \pi_X)(\bar{x} - \underline{x}) - \gamma \bar{N}_Y \big(\underbrace{\pi_{\bar{x}\bar{y}}(1 - \pi_X) - \pi_{\bar{x}\underline{y}}\pi_Y}_{=r^\Pi_{XY}}\big)(\bar{y} - \underline{y}) \\
=& a(1 - 2p_X) - \gamma\Big[\bar{N}_X\Big(a(1-a)S_X^2 + 1\Big)p_X(1 - p_X)(\bar{x} - \underline{x}) - \bar{N}_Y r^\Pi_{XY}(\bar{y} - \underline{y})\Big].
\end{aligned}
\tag{29}
$$

For $\gamma > 0$ sufficiently small, the sign of equation (29) is determined by $a(1 - 2p_X)$ and thus $q_X - p_X > 0 \iff S_X > 0$. In order to prove the second claim we show that $S_X > 0$ yields $P_X < \mu_X + \frac{1}{2}\sigma_X S_X$. From the pricing equation P_X in Theorem 1, we derive the following condition:

$$
\begin{aligned}
P_X < P_X R^f = \mu_X + aS_X\sigma_X - \gamma\bar{N}_X\Big[\sigma_X^2 + a(1-a)\sigma_X^2 S_X^2\Big] & \\
- \gamma\bar{N}_Y\Big(r_{XY}(1 - 2a) + \frac{1}{2}a\Big)\frac{2\sigma_X\sigma_Y}{\sqrt{p_X(1 - p_X)}} &< \mu_X + \frac{1}{2}\sigma_X S_X \\
\iff \quad (a - \frac{1}{2})S_X\sigma_X - \gamma\bar{N}_X\Big[\sigma_X^2 + a(1-a)\sigma_X^2 S_X^2\Big] & \\
- \gamma\bar{N}_Y\Big(r_{XY}(1 - 2a) + \frac{1}{2}a\Big)\frac{2\sigma_X\sigma_Y}{\sqrt{p_X(1 - p_X)}} &< 0,
\end{aligned}
$$

which indeed holds for $\gamma > 0$ sufficiently small and $S_X > 0$. In summary, for $S_X > 0$ we have $q_X - p_X > 0$ and $(r_{\bar{x}})^2 - (r_{\underline{x}})^2 > 0$ and thus $VP_X > 0$ as claimed.
For $S_X < 0$, the claim follows if, for $\gamma > 0$ sufficiently small, $P_X > \mu_X + \frac{1}{2}\sigma_X S_X$. We obtain:

$$
\begin{aligned}
P_X R^f = \mu_X + aS_X\sigma_X &> \Big(\mu_X + \frac{1}{2}\sigma_X S_X\Big)R^f \\
\iff& \mu_X\big(1 - R^f\big) + (a - \frac{1}{2}R^f)S_X\sigma_X > 0 \\
\iff& S_X < \frac{\mu_X\big(1 - R^f\big)}{\big(\frac{1}{2}R^f - a\big)\sigma_X} =: \bar{S}.
\end{aligned}
\tag{30}
$$

Note that, indeed, $\bar{S} < 0$ as $R^f = 1 + r^f > 1$. The second claim follows. ■

7.7 Proof of Proposition 4

By Definition 2, $SP_X = \big(q_X - p_X\big) \cdot \Big((r_{\bar{x}})^3 - (r_{\underline{x}})^3\Big)$. As the second factor is always positive, the sign is determined by $q_X - p_X$. In the proof of Proposition 3 we have shown that, for $a = 0$ and $\rho_{XY} > -\frac{\bar{N}_X \sigma_X}{\bar{N}_Y \sigma_Y}$, $q_X - p_X < 0$ and, therefore, $SP_X < 0$. The first claim follows.

We now prove the second statement. In the proof of Proposition 3 we have shown that, if $a > 0$ and $\gamma > 0$ sufficiently small, $S_X > 0$ yields $q_X - p_X > 0$ and $S_X < 0$ yields $q_X - p_X < 0$. The second claim follows. ■

7.8 Proof of Proposition 5

By the definitions of r^{Π}_{XY} and r_{XY}:

$$\text{cov}^{\Pi}\big(X, Y\big) = r^{\Pi}_{XY}(\bar{x} - \underline{x})(\bar{y} - \underline{y}), \quad \text{cov}\big(X, Y\big) = r_{XY}(\bar{x} - \underline{x})(\bar{y} - \underline{y}).$$

The claim follows if $r^{\Pi}_{XY} > r_{XY}$. From equation (18), note that:

$$\begin{aligned} r^{\Pi}_{XY} &= \pi_{\bar{x}\bar{y}} - \pi_X \pi_Y = \pi_{\bar{x}\bar{y}} - \big(\pi_{\bar{x}\bar{y}} + \pi_{\bar{x}\underline{y}}\big)\big(\pi_{\bar{x}\bar{y}} + \pi_{\underline{x}\bar{y}}\big) \\ &= \pi_{\bar{x}\bar{y}}\big(1 - \pi_{\bar{x}\bar{y}} - \pi_{\bar{x}\underline{y}} - \pi_{\underline{x}\bar{y}}\big) - \pi_{\bar{x}\underline{y}}\pi_{\underline{x}\bar{y}} = \pi_{\bar{x}\bar{y}}\pi_{\underline{x}\underline{y}} - \pi_{\bar{x}\underline{y}}\pi_{\underline{x}\bar{y}}. \end{aligned}$$

Therefore:

$$\begin{aligned} r^{\Pi}_{XY} - r_{XY} &= \pi_{\bar{x}\bar{y}}\pi_{\underline{x}\underline{y}} - \pi_{\bar{x}\underline{y}}\pi_{\underline{x}\bar{y}} - r_{XY} \\ &= \big(a + (1-2a)p_{\bar{x}\bar{y}}\big)\big(a + (1-2a)p_{\underline{x}\underline{y}}\big) - (1-2a)p_{\bar{x}\underline{y}}(1-2a)p_{\underline{x}\bar{y}} - r_{XY} \\ &= a^2 + a(1-2a)\big(p_{\underline{x}\underline{y}} + p_{\bar{x}\bar{y}}\big) + \underbrace{(1-2a)^2 p_{\bar{x}\bar{y}}p_{\underline{x}\underline{y}} - (1-2a)^2 p_{\bar{x}\underline{y}}p_{\underline{x}\bar{y}}}_{=(1-2a)^2 r_{XY}} - r_{XY} \\ &= a^2 - 2a r_{XY} + a(1-2a)\big(1 - p_X(1-p_Y) - p_Y(1-p_X)\big). \end{aligned} \tag{31}$$

The claim follows if (31) is greater than zero. All probabilities being (strictly) between zero and one yield the so-called Fréchet bounds on the dependence parameter:

$$r_{XY} < \min\left(p_X(1-p_Y), p_Y(1-p_X)\right) \text{ and } r_{XY} > \max\left(-p_X p_Y, -(1-p_X)(1-p_Y)\right). \tag{32}$$

Exploiting the Fréchet bound of equation (32), it suffices to show that:

$$a^2 - 2a\min\Big(p_X(1-p_Y), p_Y(1-p_X)\Big) + a(1-2a)\big(1 - p_X(1-p_Y) - p_Y(1-p_X)\big) > 0. \tag{33}$$

WLOG, let $p_X \leq p_Y$ (if $p_X > p_Y$, then the following arguments apply analogously due to symmetry of the algebraic expressions in p_X and p_Y). Then, $\min\big(p_X(1-p_Y), p_Y(1-p_X)\big) = p_X(1-p_Y)$ and the left-hand side of (33) becomes:

$$a^2 - 2ap_X(1-p_Y) + a(1-2a)\big(1 - p_X(1-p_Y) - p_Y(1-p_X)\big) =: f(p_X, p_Y).$$

It remains to show that $f(p_X, p_Y) > 0$. First, due to:

$$f_{p_X p_X}(p_X, p_Y) \cdot f_{p_Y p_Y}(p_X, p_Y) - f_{p_X p_Y}(p_X, p_Y)^2 < 0,$$

f has no minimum on the interior of its domain. Therefore, it remains to show that any minimum of f on the edges of its domain, which are $\big\{(0, p_Y) \mid p_Y \in [0,1]\big\}$, $\big\{(p_X, 1) \mid p_X \in [0,1]\big\}$ and $\big\{(p_X, p_X) \mid p_X \in [0,1]\big\}$, is strictly greater than zero in the former two cases and equal to zero in the latter case. Due to

$$f(0, p_Y) > 0, \iff p_Y < 1 + \frac{a}{1-2a} \text{ and } f(p_X, 1) > 0, \iff p_X > -\frac{a}{1-2a},$$

each of which is always fulfilled, any minimum of f on the first two edges must be strictly positive. Finally, turning to the third edge, it is easy to show that:

$$\begin{aligned}\bar{f}(p_X) := f(p_X, p_X) &= a^2 - 2a \cdot (p_X - p_X^2) + a(1-2a) \cdot \big(1 - 2(p_X - p_X^2)\big) \\ &= 4a(1-a) \cdot p_X^2 - 4a(1-a) \cdot p_X + a(1-a),\end{aligned}$$

has the global minimum of zero (at $p_X = p_Y = \frac{1}{2}$ and $r_{XY} = p_X(1-p_Y)$). Since any other minimum of f can only be positive, the global minimum of f is also zero. By the Fréchet bounds of equation (32), it follows that $r_{XY} < p_X(1-p_Y)$ and, therefore, $f_{p_X,p_Y} > 0$. The claim follows. ■

7.9 Proof of Proposition 6

By Definition 3 and due to bilinearity of correlation it follows that:

$$CP = \frac{q_{\bar{x}\bar{y}} - q_X q_Y}{\sqrt{q_X(1-q_X)} \cdot \sqrt{q_Y(1-q_Y)}} - \frac{p_{\bar{x}\bar{y}} - p_X p_Y}{\sqrt{p_X(1-p_X)} \cdot \sqrt{p_Y(1-p_Y)}}, \tag{34}$$

where the risk-neutral probabilities follow from (13) and are equal to:

$$\begin{aligned}
q_{\bar{x}\bar{y}} &= \pi_{\bar{x}\bar{y}}\Big(1 - \gamma\Big[\bar{N}_X(1-\pi_X)(\bar{x}-\underline{x}) + \bar{N}_Y(1-\pi_Y)(\bar{y}-\underline{y})\Big]\Big),\\
q_X &= \pi_x\Big(1 - \gamma\Big[\bar{N}_X(1-\pi_X)(\bar{x}-\underline{x}) + \bar{N}_Y\frac{r_{XY}^{\Pi}}{\pi_X}(\bar{y}-\underline{y})\Big]\Big),\\
q_Y &= \pi_y\Big(1 - \gamma\Big[\bar{N}_X\frac{r_{XY}^{\Pi}}{\pi_Y}(\bar{x}-\underline{x}) + \bar{N}_Y(1-\pi_Y)(\bar{y}-\underline{y})\Big]\Big).
\end{aligned}$$

We show that, if $\gamma = 0$ and $r_{XY} \in \big(-\frac{1}{4}, \frac{1}{4}\big)$, then $CP > 0$. Then, by continuity of the CP in γ, $CP > 0$ also holds for $\gamma > 0$ sufficiently small and $r_{XY} \in \big(-\frac{1}{4}, \frac{1}{4}\big)$ and the claim follows. Note that, $r_{XY} = \pm\frac{1}{4} \iff \rho_{XY} = \pm 1$ and it follows that $p_{\bar{x}\underline{y}} = p_{\underline{x}\bar{y}} = 0$, which is not in line with the assumption that all state probabilities are strictly positive.

For $\gamma = 0$ and $p_Y = \frac{1}{2}$, equation (34) becomes:

$$\begin{aligned}
CP &= \frac{\pi_{\bar{x}\bar{y}} - \pi_X\pi_Y}{\sqrt{\pi_X(1-\pi_X)} \cdot \sqrt{\pi_Y(1-\pi_Y)}} - \frac{p_{\bar{x}\bar{y}} - p_X p_Y}{\sqrt{p_X(1-p_X)} \cdot \sqrt{p_Y(1-p_Y)}}\\
&= \frac{2r_{XY}^{\Pi}}{\sqrt{\pi_X(1-\pi_X)}} - \frac{2r_{XY}}{\sqrt{p_X(1-p_X)}}\\
&= \frac{2\big(\frac{1}{2}a + (1-2a)r_{XY}\big)}{\sqrt{\pi_X(1-\pi_X)}} - \frac{2r_{XY}}{\sqrt{p_X(1-p_X)}}\\
&= \frac{2\big(\frac{1}{2}a + (1-2a)r_{XY}\big)}{\sqrt{p_X(1-p_X) + a(1-a)(1-2p_X)^2}} - \frac{2r_{XY}}{\sqrt{p_X(1-p_X)}}
\end{aligned}$$

$$= \frac{2\left(\frac{1}{2}a + (1-2a)r_{XY}\right)\sqrt{p_X(1-p_X)} - 2r_{XY}\sqrt{p_X(1-p_X) + a(1-a)(1-2p_X)^2}}{\sqrt{p_X(1-p_X) + a(1-a)(1-2p_X)^2} \cdot \sqrt{p_X(1-p_X)}}, \tag{35}$$

where in the third and fourth equality, equations (22) and (21) of Appendix 7.1 are used, respectively. $CP > 0$ is equivalent the nominator of equation (35) being stricly positive or, equivalently:

$$\begin{aligned} r_{XY} &< -\frac{\frac{1}{2}a\sqrt{p_X(1-p_X)}}{(1-2a)\sqrt{p_X(1-p_X)} - \sqrt{p_X(1-p_X) + a(1-a)(1-2p_X)^2}} \\ &= -\frac{\frac{1}{2}a}{1 - 2a - \sqrt{1 + a(1-a)S_X^2}} \\ &= \frac{\frac{1}{2}a}{2a + \sqrt{1 + a(1-a)S_X^2} - 1} =: h(a). \end{aligned}$$

Note that, by l'Hopital's rule:

$$\lim_{a \to 0} h(a) = \frac{\frac{1}{2}}{2 + \frac{1}{2}S_X^2} = \frac{1}{4 + S_X^2},$$

such that $h(a)$ is defined and continuous on $[0, \frac{1}{2})$. Moreover, it is easy to check that:

$$h'(a) \geq 0 \iff a^2 S_X^2 \left(\frac{1}{4}S_X^2 + 1\right) \geq 0,$$

and thus h is increasing on $[0, \frac{1}{2})$, taking its global minimum at zero. Consequently:

$$CP > 0 \iff r_{XY} < \frac{1}{4 + S_X^2}. \tag{36}$$

First, note that if $S_X = 0$, the inequality of (36) becomes $r_{XY} < \frac{1}{4}$, which is true by assumption. Second, suppose that $S_X > 0 \iff p_X < \frac{1}{2}$. Then, the Fréchet bound from inequality (32) yields $r_{XY} \leq \frac{1}{2}p_X$ and, from inequality (36), $CP > 0$ follows if:

$$\frac{1}{2}p_X < \frac{1}{4 + \frac{(1-2p_X)^2}{p_X(1-p_X)}} \iff 3p_X^2 - p_X - 1 < 0,$$

which is true for $p_X < \frac{1}{2}$. Third and similarly, if $S_X < 0 \iff p_X > \frac{1}{2}$, $CP > 0$ follows if:

$$\frac{1}{2}(1-p_X) < \frac{1}{4+\frac{(1-2p_X)^2}{p_X(1-p_X)}} \iff 3p_X^2 - 5p_X + 1 < 0,$$

which is true for $p_X > \frac{1}{2}$. The claim follows. ■

7.10 Proof of Proposition 7

Following the same lines as in the proof of Theorem 1, for $p_Y \in (0,1)$ we obtain:

$$\begin{aligned} P_X R^f = \mu_X + aS_X\sigma_X - \gamma\bar{N}_X\left[\sigma_X^2 + a(1-a)\sigma_X^2 S_X^2\right] \\ - \gamma\bar{N}_Y \frac{\sigma_X\sigma_Y}{\sqrt{p_X(1-p_X)}\sqrt{p_Y(1-p_Y)}} r_{XY}^{\Pi}, \end{aligned}$$

where $r_{XY}^{\Pi} = a(1-a) + (1-2a)r_{XY} - a(1-2a)\big(p_Y(1-p_X) + p_X(1-p_Y)\big)$. First, we calculate the derivative of the pricing equation of X with respect to skewness of Y:

$$\begin{aligned} \frac{\partial P_X R^f}{\partial p_Y} =& \gamma\bar{N}_Y \frac{\sigma_X\sigma_Y}{\sqrt{p_X(1-p_X)}\sqrt{p_Y(1-p_Y)}}\Big(a(1-2a)(1-2p_X)\Big) \\ &+ \frac{1}{2}\gamma\bar{N}_Y\Big[a(1-a) - a(1-2a)(p_X(1-p_Y) + p_Y(1-p_X)\Big] \\ &\cdot \frac{\sigma_X\sigma_Y}{\sqrt{p_X(1-p_X)}\sqrt{p_Y(1-p_Y)}} \frac{1-2p_Y}{p_Y(1-p_Y)}. \end{aligned}$$

Next, we evaluate this derivative at $p_Y = \frac{1}{2}$ and calculate the derivative with respect to the skewness of asset X:

$$\frac{\partial^2 P_X}{\partial p_Y \partial p_X} = \frac{1}{R^f} \cdot 2a(1-2a)\gamma\bar{N}_Y\sigma_X\sigma_Y \cdot \frac{\partial S_X}{\partial p_X}, \tag{37}$$

which is strictly negative as $\frac{\partial S_X}{\partial p_X} < 0$. Therefore, by continuity, the cross-derivative of (37) is also negative in a neighbourhood of $p_Y = \frac{1}{2}$. ■

7.11 The no-arbitrage condition in the Π-CAPM

In this appendix, we discuss the no-arbitrage condition of Theorem 2 in more detail. First, we prove a corollary regarding the condition and afterward we discuss the no-arbitrage condition in our benchmark calibration.

The following corollary shows that the introduction of probability weighting changes the no-arbitrage condition slightly but does not make it stronger. In particular, without (with) probability weighting, no-arbitrage holds for some left-skewed (right-skewed) assets for which the model admits arbitrage in the case with (without) probability weighting. Moreover, regardless of whether there is probability weighting or not, the boundedness of the binary assets ensures that assets of arbitrary skewness level are priced if these assets are small enough and/or risk aversion is sufficiently low. Then, all possible payoffs of the assets are part of the domain where preferences are monotone.

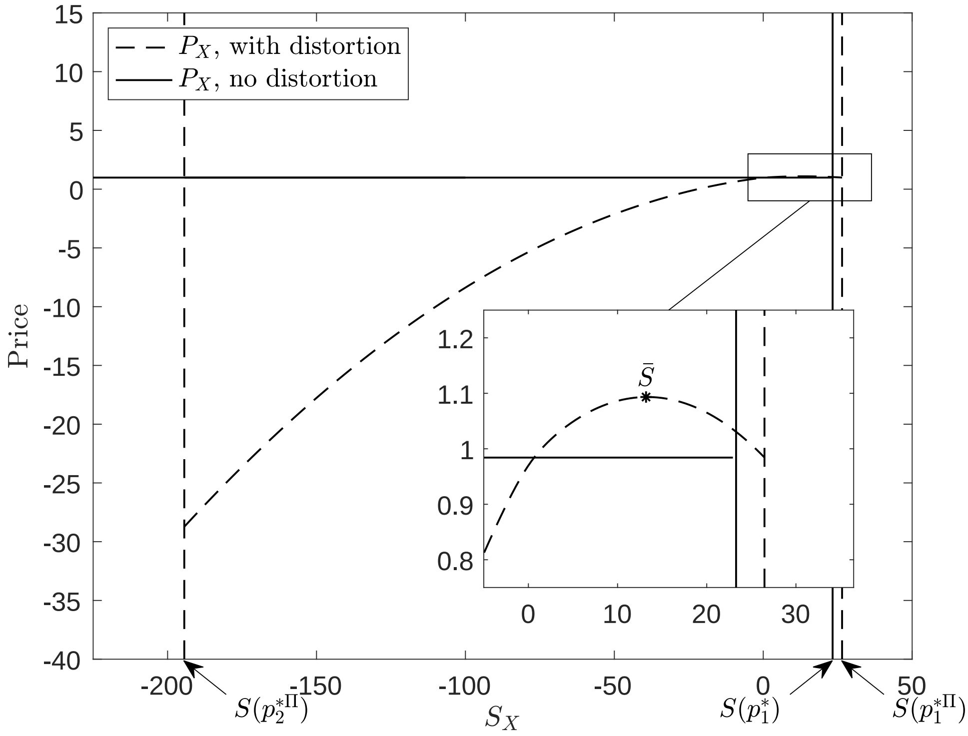
Corollary 1 (No-arbitrage in the Π-CAPM also when $p_X = 0.50$)**.** *The Π-CAPM is arbitrage-free for $p_X = 0.50$ if and only if $\bar{N}_Y\sigma_Y + \bar{N}_X\sigma_X < \frac{1}{\gamma}$ and $p_X \in (p_1^*, p_2^*)$ for $0 < p_1^* < \frac{1}{2} < p_2^* \leq 1$ as stated in the proof of Theorem 2. Moreover:*

1. *p_1^* is smallest for $a \to 0.5$ while p_2^* is largest for $a = 0$.*
2. *As $\gamma\bar{N}_X\sigma_X \to 0$, $p_1^* \to 0$ and $p_2^* \to 1$.*

Corollary 1 states that the Π-CAPM is arbitrage-free also for a symmetric X if and only if $\bar{N}_Y\sigma_Y + \bar{N}_X\sigma_X < \frac{1}{\gamma}$. Technically by Theorem 2, the no-arbitrage condition holds for $p_X \in (p_1^*, p_2^*)$ which holds by Theorem 2. By Corollary 1, this interval includes the case $p_X = 0.50$. This is an interesting case, because it is not possible for a model with MV-preferences and two assets that follow a normal distribution to be arbitrage-free as shown in Dybvig and Ingersoll (1982).

We show in Figure 13 that the no-arbitrage condition allows for extreme skewness levels of asset X. We plot the price of the skewed asset in our benchmark calibration as a function of its skewness on the interval such that the no-arbitrage condition is met.

Figure 13: The price of asset X as a function of S_X.
The figure shows the price of asset X in the Π-CAPM as a function of its skewness in the range were the no-arbitrage condition is met. The enlarged plot shows (in line with Proposition 1) that the price of the skewed asset decreases in its skewness beyond a certain skewness level $\bar{S}$, which is within the no-arbitrage region. The calibration of the Π-CAPM is from Table 1.



The dashed line in Figure 13 plots the price of the skewed asset in the case of probability distortion, while the solid line plots the price in the absence of probability distortion. The vertical dashed lines indicate the bounds of the no-arbitrage interval for our benchmark calibration with probability distortion, labelled as $S(p_1^{*\Pi})$ and $S(p_2^{*\Pi})$. The solid vertical line indicates the bound on the no-arbitrage interval in the absence of probability distortion and is labelled $S(p_1^*)$ (in the absence of probability distortion, there is no lower bound on skewness). Figure 13 has three main takeaways. First, the Π-CAPM is arbitrage-free for our benchmark calibration for a wide range of skewness levels of X. Second, the skewness intervals for which the Π-CAPM is arbitrage-free have a large overlap in the cases with and without probability distortion. In case of probability distortion, the upper bound on skewness is somewhat larger than in the absence of probability distortion, i.e. $S(p_1^{*\Pi}) > S(p_2^{*\Pi})$ which holds in general by statement 2 of Corollary 1. Third, as shown in the enlarged plot, the price of asset X decreases in its skewness beyond the skewness level $\bar{S}$, which is in line with Proposition 1. In the benchmark calibration $\bar{S} = 13.19$, which is rather large from an empirical perspective.

7.12 Empirical methodology to estimate variance and skewness premium

In this appendix, we explain our methodology to compute the variance and skewness premium, respectively. To do so, we replicate variance and skewness swaps with options. We begin with the variance swap and explain the skewness swap afterward.

A long position in the variance swap at time t with maturity T corresponds to paying an at time t agreed-upon fixed amount, the variance swap rate, and receiving the realized variance over period t to T, where the exchange occurs once and at time T. We estimate the realized variance premium of stock i at time t for maturity T as the difference between the variance swap rate and the realized variance during the lifetime of the swap. We now detail how these two components are computed. Using the approximation $r^2 \approx 2\big(e^r - 1 - r\big)$, which is mathematically convenient, the realized variance of a variance swap entered at time t with maturity T for stock i is calculated in the following way:

$$rv_{t,T}^{(i)} = \sum_{j=1}^{T} \left[2\Big(e^{r_{t+j}^{(i)}} - 1 - r_{t+j}^{(i)}\Big) \right], \tag{38}$$

where $r_{t+j}^{(i)}$ is the daily log return realized on day $t+j$ for stock i. The variance swap rate is then equal to the risk-neutral expectation of the realized variance specified in equation (38). Kozhan et al. (2013) show how to calculate the variance swap rate with maturity T for stock i at time t from option prices:

$$vs_t^{(i)} = \frac{2}{B_t} \left[\int_0^{F_t^{(i)}} \frac{P_t^{(i)}(K)}{K^2} dK + \int_{F_t^{(i)}}^{\infty} \frac{C_t^{(i)}(K)}{K^2} dK \right], \tag{39}$$

where B_t is the risk-free bond price at time t with maturity T, $F_t^{(i)}$ is the forward price of stock i at time t with maturity T, $P_t^{(i)}(K)$ and $C_t^{(i)}(K)$ are prices of European put and call options on stock i at time t with maturity T and strike price K. Note we suppressed the maturity T, because all these quantities are known at time t and because their maturity is fixed to be one month in the empirical analysis.

The prediction of a positive variance premium implies that the variance swap rate of equation (39) is larger than the expected realized variance of equation (38). Economically this means that if an investor hedges the variance risk of the underlying, she has to pay a positive premium; that is, the expected return on the variance swap is negative.

Kozhan et al. (2013) also show how to construct a skewness swap from option pricing data. The contract is similar to a variance swap in the sense that the long position holder exchanges the skewness swap rate for realized skewness at maturity. The realized skewness of a skewness swap entered at time t with maturity T for stock i is calculated as follows:

$$rs_{t,T}^{(i)} = \sum_{j=1}^{T} \left[3\Delta ve_{t+j}^{(i)} \cdot \left(e^{r_{t+j}^{(i)}} - 1\right) + 6\left(2 - 2e^{r_{t+j}^{(i)}} + r_{t+j}^{(i)} + r_{t+j}^{(i)} e^{r_{t+j}^{(i)}}\right) \right], \tag{40}$$

where $\Delta ve_{t+j}^{(i)}$ is the daily change in an entropy contract and equal to:

$$\Delta ve_{t+j}^{(i)} = ve_{t+j}^{(i)}(T-j) - ve_{t+j-1}^{(i)}(T-j+1),$$

where $ve_{t+j}^{(i)}(T-j)$ is value of the entropy contract at time $t+j$ for stock i with maturity $T-j$.

Kozhan et al. (2013) show that this version of the skewness swap is similar to a specification where the floating rate is equal to the sum of cubic daily returns, but analytically more tractable. The skewness swap rate is equal to the risk-neutral expectation of the realized skewness in equation (40). Like the variance swap rate, the skewness swap rate is computed from option prices. Kozhan et al. (2013) show that the skewness swap rate is equal to the difference between the variance swap rate of equation (39) and the entropy contract with maturity T for company i at time t and is defined as:

$$ve_t^{(i)} = \frac{2}{B_t} \left[\int_0^{F_t^{(i)}} \frac{P_t^{(i)}(K)}{K \cdot F_t^{(i)}} dK + \int_{F_t^{(i)}}^{\infty} \frac{C_t^{(i)}(K)}{K \cdot F_t^{(i)}} dK \right].$$

The skewness swap rate is then defined as follows:

$$s_t^{(i)} = 3\left(ve_t^{(i)} - vs_t^{(i)}\right). \tag{41}$$

The Π-CAPM predicts that the skewness premium of stock i at time t, defined as the difference between the skewness swap rate of equation (41) and the expected realized skewness of equation (40), is positive for right-skewed assets and negative for left-skewed assets. This means that an investor who takes a long position in a skewness swap for a left-skewed (right-skewed) asset receives (pays) a premium.

7.13 Replication and extension of sample period of Kozhan et al. (2013)

In this appendix, we first discuss the results of our methodology over the same sample period as in Kozhan et al. (2013). We show that we match the results very closely and, therefore, this analysis serves as a validation of our methodology. Afterward, we extend the sample period of Kozhan et al. (2013) by six years and show that the results continue to hold.

The results are shown in Table 4 and correspond to Panel A of Table 1 of Kozhan et al. (2013), where

$$xv_{t,T} = \frac{rv_{t,T}}{vs_t} - 1 \text{ and } xs_{t,T} = \frac{rs_{t,T}}{s_t} - 1$$

correspond to realized returns on monthly S&P 500 variance and skewness swaps.[29] Note that, with these definitions, the negative of the respective swap return corresponds to how we defined the variance and skewness premiums. We also dropped the superscript i, because in the following we discuss the results for the S&P 500 only). Furthermore, $rskew_{t,T}$ is defined as follows:

$$rskew_{t,T} = \frac{rs_{t,T}}{\left(vs_t\right)^{\frac{3}{2}}}.$$

[29] Our definition in the main text follows Bollerslev et al. (2009) and corresponds to receiving the fixed leg of the variance swap, leading to a positive variance premium. The definition in Kozhan et al. (2013) corresponds to receiving the floating leg, leading to a negative risk premium.

Table 4: The table shows descriptive statistics of the variance and skewness swap with the S&P 500 as underlying. Sample period is from 01-1996 to 12-2011.

Variable	Mean	SD	25%	Median	75%
$vs \times 100$	0.476	0.450	0.234	0.363	0.552
$rv \times 100$	0.365	0.590	0.108	0.207	0.377
xv	-0.273	0.663	-0.591	-0.437	-0.207
S	-1.302	0.422	-1.550	-1.291	-1.012
$rskew$	-0.685	1.148	-0.679	-0.375	-0.241
xs	-0.442	0.981	-0.815	-0.699	-0.432

Similar to Kozhan et al. (2013) we focus on the sample period from 1996 to 2012. Overall, we match the results of Kozhan et al. (2013) on the components of the variance premium and skewness premium of the S&P 500 closely. Small differences in the distribution can be driven by the fact that Kozhan et al. (2013) only consider one swap each month, whereas our data allows us to consider one swap each trading day. Further, from Table 4 it follows that our methodology underestimates the risk-neutral skewness S and realized skewness $rskew$ of the S&P 500. This underestimation is likely driven by that fact that our extrapolation is not able to fully capture the steep volatility smile of S&P 500 options. The effect of this underestimation is the same in the calculation of the risk-neutral skewness S and realized skewness $rskew$, and, therefore, we do match the return on the skewness swap xs closely. The correlation between the return on a variance swap xv and skewness swap xs equals 0.858 and is similar to the correlation reported Kozhan et al. (2013), which is 0.897.

We now extend the sample period from Kozhan et al. (2013) by six years and show that it yields similar results, which we present in Table 5.

Table 5: The table shows descriptive statistics of the variance and skewness swap with the S&P 500 as underlying. Sample period is from 01-2012 to 12-2017.

Variable	Mean	SD	25%	Median	75%
$vs \times 100$	0.206	0.105	0.141	0.178	0.244
$rv \times 100$	0.122	0.108	0.055	0.090	0.158
xv	-0.395	0.546	-0.668	-0.523	-0.285
S	-2.002	0.448	-2.255	-1.965	-1.706
$rskew$	-0.797	1.423	-0.837	-0.450	-0.284
xs	-0.594	0.751	-0.852	-0.762	-0.555

The realized returns on the variance and skewness swap over the period 01-2012 to 12-2017 are lower than those for the sample period of Kozhan et al. (2013). This makes sense as realized returns are large during periods of large volatility and the sample period of Kozhan et al. (2013) includes the financial crisis and the tech bubble. The correlation between the realized returns on variance swaps and skewness swaps is similar as in the early sample period: 0.868.