Steven Heston: Recovering the Variance Premium

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WHAT IS THE RECOVERY PROBLEM?

Using observed cross-section(s) of prices (of Arrow–Debreu securities), infer …

- preference parameters
- investors' beliefs

imposing 'as little structure as possible'. Only:

- Markovianity
- time invariance
- minimal restrictions on preferences

This is an identification problem.

A FINITE STATE SPACE FRAMEWORK

Physical environment

- X — a discrete-time stationary and ergodic Markov chain with n states

Investor beliefs and preferences

- $\mathbf{P} = [p_{ij}]$ — transition matrix — (subjective) beliefs

$$p_{ij} = P\left(X_{t+1} = j \mid X_t = i\right)$$

- $\mathbf{M} = [m_{ij}]$ — stochastic discount factor

$$m_{ij} \text{ — state-specific discount rate between states } i \text{ and } j$$

Asset prices

- $\mathbf{Q} = [q_{ij}]$ — matrix of prices of one-period Arrow securities

$$q_{ij} \text{ — price in state } i \text{ of one unit of state-}j \text{ cash flow next period}$$

AN IDENTIFICATION PROBLEM

Arrow prices encode both beliefs and preferences:

$$q_{ij} = p_{ij} m_{ij}$$

Suppose we observe asset prices $[q_{ij}]$.

- Identification problem: Can we separately identify $[p_{ij}]$ and $[m_{ij}]$?

$$\underbrace{q_{ij}}_{n \times n \text{ equations}} = \underbrace{p_{ij}}_{n \times (n-1) \text{ unknowns}} \underbrace{m_{ij}}_{n \times n \text{ unknowns}}$$

Underidentification!!!

PERRON–FROBENIUS THEORY

Let $\mathbf{Q}$ be a matrix with strictly positive entries. Then there exists

- a unique strictly positive eigenvector e
- associated with the largest eigenvalue $\exp(\eta)$:

$$\mathbf{Q}e = \exp(\eta)\, e$$

Hence, given asset prices $\mathbf{Q}$, we can back out e and $\exp(\eta)$.

- What can we do with them?

A 'LONG-RUN PRICING' PROBABILITY MEASURE

- Use the results from the Perron–Frobenius problem to construct

$$\widetilde{p}_{ij} = \exp(-\eta)\, q_{ij} \frac{e_j}{e_i}$$

- $\widetilde{\mathsf{P}} = [\widetilde{p}_{ij}]$ is a transition matrix (rows sum up to one)

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$$q_{ij} = \underbrace{\exp\left(\eta\right) \frac{e_i}{e_j}}_{\widetilde{m}_{ij}} \widetilde{p}_{ij}$$

- There is no claim that $\widetilde{\mathsf{M}} = \left[\widetilde{m}_{ij}\right]$ is the true stochastic discount factor or that $\widetilde{\mathsf{P}} = \left[\widetilde{p}_{ij}\right]$ represents investors' beliefs.

Definition
The pair $(\mathbf{M}, \mathbf{P})$ explains asset prices $\mathbf{Q}$ if $q_{ij} = p_{ij} m_{ij}$ for every i, j.

- Take **any** random variable $\mathbf{H} = [h_{ij}]$ with mean one: $\sum_{j=1}^{n} h_{ij} p_{ij} = 1$.
- Define

$$
\begin{aligned}
\mathbf{P}^H &= \left[p_{ij}^H\right] \quad \text{with } p_{ij}^H = h_{ij} p_{ij} \\
\mathbf{M}^H &= \left[m_{ij}^H\right] \quad \text{with } m_{ij}^H = \frac{m_{ij}}{h_{ij}}
\end{aligned}
$$

Then $\mathbf{P}^H$ is a valid transition matrix and $\left(\mathbf{M}^H, \mathbf{P}^H\right)$ also explains asset prices $\mathbf{Q}$.

How are S and $\widetilde{\mathsf{S}}$ related?

$$q_{ij} = \underbrace{\exp(\eta)\frac{e_i}{e_j}}_{\widetilde{s}_{ij}}\widetilde{p}_{ij}$$

DECOMPOSITION OF THE STOCHASTIC DISCOUNT FACTOR

How are S and $\widetilde{\text{S}}$ related?

$$q_{ij} = \underbrace{\exp(\eta)\,\frac{e_i}{e_j}}_{\widetilde{S}_{ij}}\widetilde{p}_{ij} = \underbrace{\exp(\eta)\,\frac{e_i}{e_j}\,\overbrace{\frac{\widetilde{p}_{ij}}{p_{ij}}}^{}}_{S_{ij}}\, p_{ij}$$

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  - multiplicative martingale increment (change of measure).

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A unique decomposition: Every stochastic discount factor in this environment has to have this form.

- A deterministic drift $\exp(\eta)$.
- A stationary component e_i/e_j.
- A martingale component $\widetilde{h}_{ij}$.

WHAT IS IDENTIFIED?

$$q_{ij} = \underbrace{\exp(\eta)\frac{e_i}{e_j}}_{\widetilde{s}_{ij}}\widetilde{p}_{ij}$$

Given **Q**, we identified

- the eigenfunction-eigenvalue pair $(e, \eta) \implies$ pair $(\widetilde{S}, \widetilde{P})$

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What remains unidentified?

- the decomposition $\widetilde{p}_{ij} = \widetilde{h}_{ij} p_{ij} \implies$ pair (S, P)
- i.e., so far we learned nothing about P

HOW TO RECOVER INVESTORS' BELIEFS?

We must impose economic restrictions on the martingale component $\mathbf{H}$.

- Ross (2015): $\mathbf{H} = 1$. This implies $P = \widetilde{P}$.

Theory: $\mathbf{H} \neq 1$ and volatile

- recursive preferences
- consumption with stochastic growth

Empirics: $\mathbf{H} \neq 1$ and volatile

- time series data + imposing rational expectations
- tests reject $\mathbf{H} = 1$ in broad stock markets and bond markets
  - Alvarez and Jermann (2005), Qin, Linetsky and Nie (2016), Bakshi, Chabi-Yo and Gao (2016), Audrino, Huitema and Ludwig (2016), ...

Additional mathematical complications

- The counterpart to the eigenproblem $\mathbf{Q}e = \exp(\eta)\, e$ does not generally have a unique solution.
- A unique recovered probability measure $\widetilde{P}$ that preserves stationarity and ergodicity of X_t:

$$\frac{S_{t+j}}{S_t} = \underbrace{\exp(\eta j)}_{\text{deterministic trend}} \quad \underbrace{\frac{e(X_t)}{e(X_{t+j})}}_{\text{stationary component}} \quad \underbrace{\frac{H_{t+j}}{H_t}}_{\text{martingale}}$$

  - Hansen and Scheinkman (2009), Borovička, Hansen and Scheinkman (2016)

This does not address in any way the identification of H and hence of investors' beliefs P from cross-sectional asset price data Q.

A SQUARE-ROOT PROCESS EXAMPLE

Consider state variable X_t following the square-root process[1]

$$dX_t = -\kappa\left(X_t - \mu\right)dt + \sigma\sqrt{X_t}dW_t$$

and prices generated by the 'true' stochastic discount factor

$$d\log M_t = \beta dt - \frac{1}{2}\alpha^2 X_t dt + \alpha\sqrt{X_t}dW_t$$

- risk-free rate $-\beta$
- price of variance risk $-\alpha\sqrt{X_t}$
- risk-neutral price dynamics of X_t

$$dX_t = -\kappa_n\left(X_t - \mu\kappa/\kappa_n\right) + \sigma\sqrt{X_t}dW_t^*$$

with $\kappa_n = \kappa - \sigma\alpha$.

[1] Borovička, Hansen, Scheinkman (2016), Example 4

RECOVERY

In general infinitely many strictly positive eigenfunctions.

- Consider those of the form

$$e(x) = \exp(\nu x).$$

- Two solutions

$$\nu = 0 \qquad \nu = \frac{2(\kappa - \alpha\sigma)}{\sigma^2}$$

Which one to pick?

- Borovička, Hansen, Scheinkman (2016): At most one solution is such that the recovered probability measure preserves stationarity and ergodicity.
- This provides a unique way of decomposing M.
- But does not in any way address the problem of identifying H from Q.
- In fact, neither ν above recovers the original dynamics!

HESTON (2017)

A ‘standard’ estimation approach

Impose parameterized physical and risk-neutral dynamics

- Infer stochastic discount factor

$$M_t = S_t^{\gamma} \exp\left(\beta t + \xi v_t + \eta \int_0^t v_s ds\right)$$

- Dislike the martingale arising from $\int_0^t v_s ds \doteq Y_t$.
    - martingale in S_t is fine

Impose a restriction $\eta = 0$ (not needed here!)

- Given risk-neutral measure (prices), this restricts the physical measure.

Estimate parameters γ, β, ξ using time-series information.

CONNECTION TO RECOVERY

Original recovery problem

- Impose transition independence in stationary state variables.
- Here, v_t is the only natural stationary state variable.
- Solution (given fixed r): $\gamma = \eta = \xi = 0 \implies$ risk-neutrality.

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This paper allows transition independence in nonstationary S_t, too.

- New state variable $S_t \implies$ vastly expands the set of solutions.
- 'Anything goes': e.g., M_t is transition independent in itself.
- Or use $Y_t = \int_0^t v_s ds$ as another state variable.
- See Borovička, Hansen and Scheinkman, Section 7.

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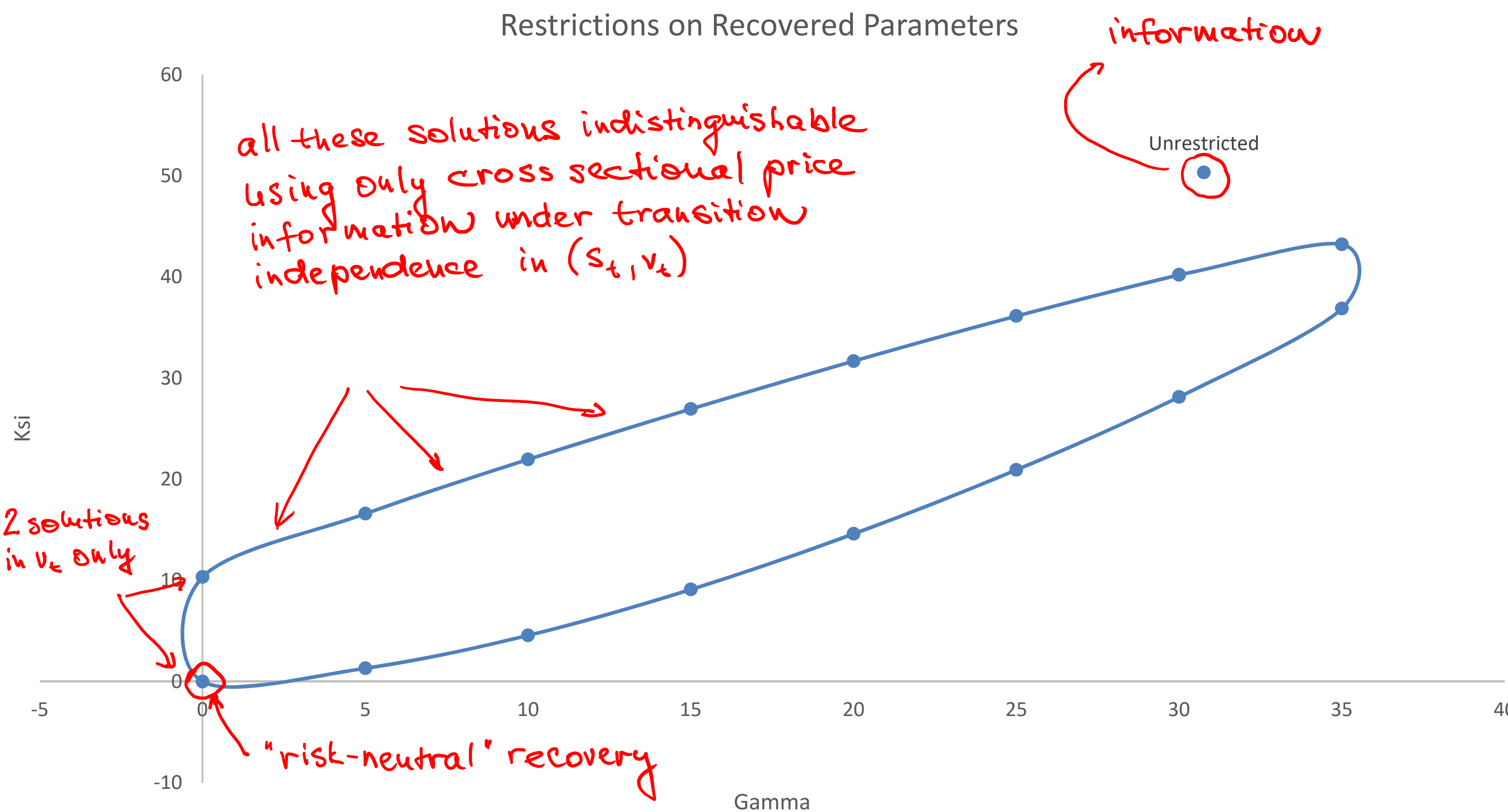
This makes the identification problem much worse.

- For conceptually different reasons than the 'misspecified recovery'.
- Still unable to pin down H_t from cross-sectional data.

Recovery Restrictions



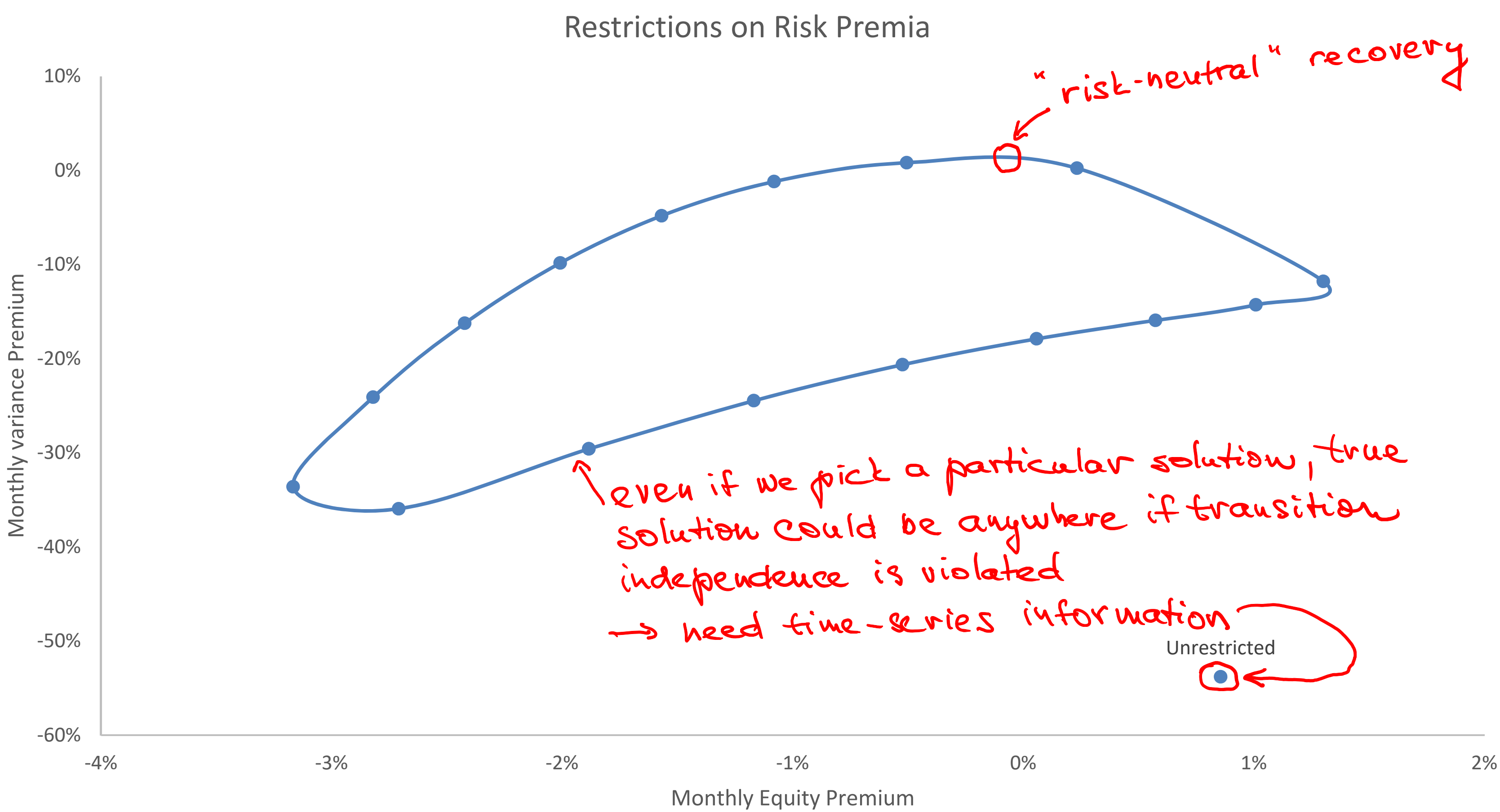
identification from time-series information

all these solutions indistinguishable using only cross sectional price information under transition independence in (s_t, v_t)

2 solutions in v_t only

"risk-neutral" recovery

Restrictions on Risk Premia



"risk-neutral" recovery

even if we pick a particular solution, true solution could be anywhere if transition independence is violated

→ need time-series information

TAKEAWAYS

'Recovery' using cross-sectional information

- can pin down transitory component of SDF
- cannot pin down investors' beliefs P without additional assumptions
- using non-stationary variables as states makes the identification problem worse

What do we need

- time-series information
- economically motivated 'structural' restrictions on the form of SDF

And this is exactly what this paper aims for.