Steven Heston: Recovering the Variance Premium

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WHAT IS THE RECOVERY PROBLEM?

Using observed cross-section(s) of prices (of Arrow–Debreu securities), infer...

- preference parameters
- investors’ beliefs

imposing ‘as little structure as possible’. Only:

- Markovianity
- time invariance
- minimal restrictions on preferences

This is an identification problem.
A FINITE STATE SPACE FRAMEWORK

Physical environment

- $X$ — a discrete-time stationary and ergodic Markov chain with $n$ states

Investor beliefs and preferences

- $P = [p_{ij}]$ — transition matrix — (subjective) beliefs
  \[ p_{ij} = P(X_{t+1} = j \mid X_t = i) \]
- $M = [m_{ij}]$ — stochastic discount factor
  \[ m_{ij} \] — state-specific discount rate between states $i$ and $j$

Asset prices

- $Q = [q_{ij}]$ — matrix of prices of one-period Arrow securities
  \[ q_{ij} \] — price in state $i$ of one unit of state-$j$ cash flow next period
Arrow prices encode both beliefs and preferences:

\[ q_{ij} = p_{ij} m_{ij} \]

Suppose we observe asset prices \([q_{ij}]\).

- Identification problem: Can we separately identify \([p_{ij}]\) and \([m_{ij}]\)?

\[
\begin{align*}
\underbrace{q_{ij}}_{n \times n \text{ equations}} &= \underbrace{p_{ij}}_{n \times (n - 1) \text{ unknowns}} \underbrace{m_{ij}}_{n \times n \text{ unknowns}}
\end{align*}
\]

Underidentification!!!
Let $Q$ be a matrix with strictly positive entries. Then there exists

- a unique **strictly positive eigenvector** $e$
- associated with the **largest eigenvalue** $\exp(\eta)$:

$$Qe = \exp(\eta)e$$

Hence, given asset prices $Q$, we can back out $e$ and $\exp(\eta)$.

- What can we do with them?
A ‘LONG-RUN PRICING’ PROBABILITY MEASURE

• Use the results from the Perron–Frobenius problem to construct

$$\tilde{p}_{ij} = \exp (-\eta) q_{ij} \frac{e_j}{e_i}$$

• $\tilde{P} = [\tilde{p}_{ij}]$ is a transition matrix (rows sum up to one)
• Use the results from the Perron–Frobenius problem to construct

\[ \tilde{p}_{ij} = \exp(-\eta) \frac{e_j}{e_i} q_{ij} \]

• \( \tilde{P} = [\tilde{p}_{ij}] \) is a transition matrix (rows sum up to one)

• Invert to obtain a decomposition

\[ q_{ij} = \exp(\eta) \frac{e_i}{e_j} \tilde{p}_{ij} \]

\[ \tilde{m}_{ij} \]
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- \( \tilde{P} = [\tilde{p}_{ij}] \) is a transition matrix (rows sum up to one)
- Invert to obtain a decomposition

\[ q_{ij} = \exp(\eta) \frac{e_i}{e_j} \tilde{p}_{ij} \]

\[ \tilde{m}_{ij} \]

- There is no claim that \( \tilde{M} = [\tilde{m}_{ij}] \) is the true stochastic discount factor or that \( \tilde{P} = [\tilde{p}_{ij}] \) represents investors’ beliefs.
Definition
The pair \((M, P)\) explains asset prices \(Q\) if \(q_{ij} = p_{ij}m_{ij}\) for every \(i, j\).

- Take any random variable \(H = [h_{ij}]\) with mean one: \(\sum_{j=1}^{n} h_{ij}p_{ij} = 1\).
- Define

\[
\begin{align*}
  P^H &= \begin{bmatrix} p^H_{ij} \end{bmatrix} \quad \text{with} \quad p^H_{ij} = h_{ij}p_{ij} \\
  M^H &= \begin{bmatrix} m^H_{ij} \end{bmatrix} \quad \text{with} \quad m^H_{ij} = \frac{m_{ij}}{h_{ij}}
\end{align*}
\]

Then \(P^H\) is a valid transition matrix and \((M^H, P^H)\) also explains asset prices \(Q\).
How are $S$ and $\tilde{S}$ related?

$$q_{ij} = \exp(\eta) \frac{e_i}{e_j} \tilde{p}_{ij}$$

$\tilde{S}_{ij}$
How are $S$ and $\tilde{S}$ related?

$$q_{ij} = \exp(\eta) \frac{e_i}{e_j} \tilde{p}_{ij} = \exp(\eta) \frac{e_i}{e_j} \frac{\tilde{p}_{ij}}{p_{ij}} \frac{p_{ij}}{\bar{p}_{ij}}$$

$$\tilde{S}_{ij} \quad S_{ij}$$
How are $S$ and $\tilde{S}$ related?

$$q_{ij} = \exp(\eta) \frac{e_i}{e_j} \tilde{p}_{ij} = \exp(\eta) \frac{e_i}{e_j} \frac{\tilde{p}_{ij}}{p_{ij}} p_{ij}$$
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$q_{ij} = \exp(\eta) \frac{e_i}{e_j} \tilde{p}_{ij} = \exp(\eta) \frac{\tilde{p}_{ij}}{p_{ij}} p_{ij}$

- $[\tilde{h}_{ij}]$ has conditional expectation equal to one
  - multiplicative martingale increment (change of measure).
How are $S$ and $\tilde{S}$ related?

\begin{equation}
q_{ij} = \exp(\eta) \frac{e_i}{e_j} \tilde{p}_{ij} = \exp(\eta) \frac{e_i}{e_j} \frac{\tilde{p}_{ij}}{p_{ij}} p_{ij}
\end{equation}

- $[\tilde{h}_{ij}]$ has conditional expectation equal to one
  - multiplicative martingale increment (change of measure).

**A unique decomposition**: Every stochastic discount factor in this environment has to have this form.

- A deterministic drift $\exp(\eta)$.
- A stationary component $e_i/e_j$.
- A martingale component $\tilde{h}_{ij}$. 
Given $Q$, we identified

- the eigenfunction-eigenvalue pair $(e, \eta) \mapsto$ pair $(\tilde{S}, \tilde{P})$
Given $Q$, we identified

- the eigenfunction-eigenvalue pair $(e, \eta) \implies$ pair $(\tilde{S}, \tilde{P})$

What remains unidentified?

- the decomposition $\tilde{p}_{ij} = \tilde{h}_{ij} p_{ij} \implies$ pair $(S, P)$
- i.e., so far we learned nothing about $P$
HOW TO RECOVER INVESTORS’ BELIEFS?

We must impose economic restrictions on the martingale component $H$.

- **Ross (2015)**: $H = 1$. This implies $P = \tilde{P}$.

**Theory**: $H \neq 1$ and volatile

- recursive preferences
- consumption with stochastic growth

**Empirics**: $H \neq 1$ and volatile

- time series data + imposing rational expectations
- tests reject $H = 1$ in broad stock markets and bond markets
  - Alvarez and Jermann (2005), Qin, Linetsky and Nie (2016), Bakshi, Chabi-Yo and Gao (2016), Audrino, Huiitema and Ludwig (2016), ...
CONTINUOUS TIME / CONTINUOUS STATE SPACE

Additional mathematical complications

- The counterpart to the eigenproblem $Qe = \exp(\eta)e$ does not generally have a unique solution.
- A unique recovered probability measure $\tilde{P}$ that preserves stationarity and ergodicity of $X_t$:

$$\frac{S_{t+j}}{S_t} = \exp(\eta j)$$

- Deterministic trend

$$\frac{e(X_t)}{e(X_{t+j})}$$

- Stationary component

$$\frac{H_{t+j}}{H_t}$$

- Martingale

- Hansen and Scheinkman (2009), Borovička, Hansen and Scheinkman (2016)

This does not address in any way the identification of $H$ and hence of investors’ beliefs $P$ from cross-sectional asset price data $Q$. 
Consider state variable $X_t$ following the square-root process\(^1\)

$$dX_t = -\kappa (X_t - \mu) \, dt + \sigma \sqrt{X_t} \, dW_t$$

and prices generated by the ‘true’ stochastic discount factor

$$d \log M_t = \beta dt - \frac{1}{2} \alpha^2 X_t dt + \alpha \sqrt{X_t} \, dW_t$$

- risk-free rate $-\beta$
- price of variance risk $-\alpha \sqrt{X_t}$
- risk-neutral price dynamics of $X_t$

$$dX_t = -\kappa_n (X_t - \mu \kappa / \kappa_n) + \sigma \sqrt{X_t} \, dW^*_t$$

with $\kappa_n = \kappa - \sigma \alpha$.

\(^1\)Borovička, Hansen, Scheinkman (2016), Example 4
In general infinitely many strictly positive eigenfunctions.

- Consider those of the form
  \[ e(x) = \exp(\nu x) . \]

- Two solutions
  \[ \nu = 0 \quad \nu = \frac{2(\kappa - \alpha \sigma)}{\sigma^2} . \]

Which one to pick?

- **Borovička, Hansen, Scheinkman (2016):** At most one solution is such that the recovered probability measure preserves stationarity and ergodicity.
  
- This provides a unique way of decomposing \( M \).
  
- But does not in any way address the problem of identifying \( H \) from \( Q \).
  
- In fact, neither \( \nu \) above recovers the original dynamics!
A ‘standard’ estimation approach

Impose parameterized physical and risk-neutral dynamics

- Infer stochastic discount factor

\[ M_t = S_t^\gamma \exp \left( \beta t + \xi v_t + \eta \int_0^t v_s ds \right) \]

- Dislike the martingale arising from \( \int_0^t v_s ds \equiv Y_t \).
  - martingale in \( S_t \) is fine

Impose a restriction \( \eta = 0 \) (not needed here!)

- Given risk-neutral measure (prices), this restricts the physical measure.

Estimate parameters \( \gamma, \beta, \xi \) using time-series information.
Original recovery problem

- Impose transition independence in stationary state variables.
- Here, $v_t$ is the only natural stationary state variable.
- Solution (given fixed $r$): $\gamma = \eta = \xi = 0 \implies$ risk-neutrality.
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- Solution (given fixed \( r \)): \( \gamma = \eta = \xi = 0 \rightarrow \text{risk-neutrality} \).

This paper allows transition independence in nonstationary \( S_t \), too.

- New state variable \( S_t \) \( \rightarrow \) vastly expands the set of solutions.
- ‘Anything goes’: e.g., \( M_t \) is transition independent in itself.
- Or use \( Y_t = \int_0^t v_s ds \) as another state variable.
- See Borovička, Hansen and Scheinkman, Section 7.
Original recovery problem

- Impose transition independence in **stationary** state variables.
- Here, $\nu_t$ is the only natural stationary state variable.
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This paper allows transition independence in **nonstationary** $S_t$, too.

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This makes the **identification problem much worse**.

- For conceptually different reasons than the ‘misspecified recovery’.
- Still unable to pin down $H_t$ from cross-sectional data.
Recovery Restrictions

Restrictions on Recovered Parameters

- All these solutions indistinguishable using only cross-sectional price information under transition independence in (s_t, v_t)
- Identification from time-series information
- 2 solutions in v_t only
- "risk-neutral" recovery

Unrestricted
Restrictions on Risk Premia

Even if we pick a particular solution, true solution could be anywhere if transition independence is violated → need time-series information.

"risk-neutral" recovery
‘Recovery’ using cross-sectional information

- can pin down transitory component of SDF
- cannot pin down investors’ beliefs $P$ without additional assumptions
- using non-stationary variables as states makes the identification problem worse

What do we need

- time-series information
- economically motivated ‘structural’ restrictions on the form of SDF

And this is exactly what this paper aims for.