Recovering the Variance
Premium

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Preliminary
Ross (2015) Recovery:

• Recover “physical” probabilities from options.

• Limitations:
  – Requires a stationary state space.
  – Too good to be true (binomial or Black-Scholes).

• Relies on interest rate variation.
  – Constant interest rates just recover risk-neutrality.
    • Predicts that forward measure is risk-neutral.
    • Predicts long bond is log-efficient.
    • These predictions are obviously false.
New Generalized Recovery

  – Log(pricing kernel) can be cointegrated with stock.
• New restrictions between equity premium and variance premium.
  – A long “power security” is log-efficient.
• Measure and test the variance premium.
Heston (1993) Model

• Risk-neutral dynamics:
  \[dS = rSdt + \sqrt{\nu}Sdz_1^*, \quad (1)\]
  \[dv = \kappa^*(\theta^*-\nu)dt + \sigma\sqrt{\nu}Sdz_2^*.\]

• Observable “physical” dynamics:
  \[dS = (r+\mu\nu)Sdt + \sqrt{\nu}Sdz_1, \quad (2)\]
  \[dv = \kappa(\theta-\nu)dt + \sigma\sqrt{\nu}dz_2.\]

• Martingale Condition:
  \[U(t)M(t) = E_t[U(t+\Delta)M(t+\Delta)].\]
What is $M(t)$?

- Proposition 1:

  Risk-neutral (1) and physical dynamics (2) imply a unique $M(t)$.

  
  
  $M(t) = S(t) \exp(\beta t + \eta \int_0^t \nu(s)ds + \xi \nu(t))$.

- Solve or invert the interest rate $r$, equity premium $\mu$ and variance premium $\kappa^* - \kappa$ in terms of $\beta$, $\gamma$, $\eta$, and $\xi$.
  - Impose economic restrictions.
  - I hate that path-dependent $\eta$ term!
Merton’s (1973) Bucket Shop Assumption

• Bucket Shop Assumption on option value:
  \[ U(t) = U(S(t), v(t), t). \]

• Ross’s Transition-Independence Assumption:
  \[ M(t) = M(S(t), v(t), t) = e^{\beta t} h(S(t), v(t)). \]
  Price kernel should depend on where we are, not how we got there (through diffusion, jumps, etc.).

• \( M(t) \) should not depend on \( \int_0^t v(s)ds \).
  – The state space \( \{S(t), v(t)\} \) should be enough.
  – Habit persistence could be incorporated into current state variables.
Path-Independence

• Constant rate of time preference $\beta$.
• $M$ should be homogeneous in $S(t)$.
  – Returns do not depend on level of $S(t)$.
  – Options depend on moneyness, not level of $S(t)$.

• $M(t) = e^{\beta t} S(t)^\gamma h(v(t)),$

where reciprocal marginal utility $N(v) = 1/h(v)$ satisfies the P.D.E. of Linetsky and Qin (2016):

$$\frac{1}{2} \sigma^2 v N'' + \left[ \kappa^* (\theta^* - v) - \rho \sigma v \right] N' + \frac{1}{2} \gamma (\gamma + 1) v - \beta - (\gamma + 1) r \right] N = 0.$$
Recovery Theorem

• Given $\gamma$ and risk-neutral dynamics (1), Proposition 1 shows all path-independent pricing kernels that give stationary physical dynamics.

$$h(v) = e^{\xi v(t)},$$

where $\xi > 0$ satisfies a quadratic equation to make $\eta = 0$. If $\gamma < 0$, then there is only one positive root.

• We have recovered the physical dynamics (2).

• Does not recover the mean in Black-Scholes unless you know $\gamma$.

• This works in more general models.
Valuation of a “Power” Security

• P.D.E.:
\[ \frac{1}{2} \nu S^2 U_{ss} + \rho \sigma v S U_{sv} + \frac{1}{2} \sigma^2 v U_{ss} + r S U_s + \kappa^* (\theta^* - v) U_v - r U + U_t = 0. \]

• Terminal Payoff:
\[ U(S,v,t; \phi, T) = S(T)^\phi. \]

• Solution:
\[ U(S,v,t; \phi, T) = S(t)^\phi e^{C(T-t) + D(T-t)v(t)}, \]
where \( C(.) \) and \( D(.) \) are complicated.
Long-Term Power Security

• When $\phi = -\gamma$, $D(\infty) \rightarrow -\xi$.
  
  – Long-term option prices reveal variance preference!

\[
U(S(t), v(t), t; -\gamma, T) = 
\]

\[
E_t \left[ \frac{U(S(T), v(T), T; -\gamma, T) M(S(T), v(T), T)}{M(S(t), v(t), t)} \right].
\]
Model-Free Test

- When $\phi = -\gamma$, the gross return on a long Power Security is the reciprocal of marginal rate of substitution.

$$ R_\infty(t+\Delta) \equiv \lim_{T \to \infty} \frac{U(S(t+\Delta),v(t+\Delta),t+\Delta; -\gamma,T)}{U(S(t),v(t),t; -\gamma,T)} \frac{M(S(t),v(t),t)}{M(S(t+\Delta),v(t+\Delta),t+\Delta)}. $$

- i.e., the long-term Power Security is growth optimal.
- Use Breeden-Litzenberger to construct power security from vanilla options.
- This even works when the Power Security uses a proxy $S^*$, as long as $\log(S^*)$ is cointegrated with $\log(S)$, which is cointegrated with $\log(M)$. 
Estimating the Variance Premium

• If $\gamma, \rho < 0$, then the model predicts a positive equity premium and negative variance premium.

• Two strategies (which nobody reports!):
  
  – Monthly $\text{VIX}^2$ portfolio.
    • Adjust for exact number of days in trading month.
    • Portfolio has log-payoff,
    • 99% correlated with variance swap.

  – Bimonthly $\text{VXV}^2$ portfolio.
    • Buy the 3-month $\text{VXV}^2$ portfolio,
    • sell 2 months later using the $\text{VIX}^2$ price.
    • 99% correlated with two-month variance swap + $\text{VIX}^2$. 
Monthly Data

• CRSP risk-free T-bill return.
• CBOE S&P 500 Total Return Index.
• VIX 1990-2016 (27 years).
• VXV 2008-2016 (only 9 years).
Option Volatility Indices Are 99% Correlated
# Monthly Summary Statistics, 1990-2016 (VXV is 2008-2016)

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<th>Summary Statistics of Monthly Data, 1990-2016</th>
<th>Correlations</th>
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<tr>
<td></td>
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GMM Restrictions on Gross Returns $R_i(t)$ and Excess Returns

- Average (unconditional) equity premium:
  $$E[(R_{S&P}(t)-R_f(t))M(t)] = 0.$$ 

- Average variance premium:
  $$E[(R_{VIX}(t)-R_f(t))M(t)] = 0.$$ 

- Average risk-free return (gives $\beta$):
  $$E[R_f(t)M(t)] = 1.$$ 

- Conditional risk-free return:
  $$E[VIX^2(t)(R_f(t+\Delta)M(t+\Delta)-1)] = 0,$$
  or
  $$\text{Cov}[VIX^2(t), R_f(t+\Delta)M(t+\Delta)] = 0.$$
Recovery Restrictions

Restrictions on Recovered Parameters

- Unrestricted
Restrictions on Risk Premia

Monthly variance Premium

Monthly Equity Premium

Restrictions on Risk Premia

Unrestricted
Conclusion

• The pricing kernel $M(t)$ should jointly explain the cross-section of returns and the conditional predicted level.

• GMM does not reject with three parameters $(\beta, \gamma, \xi)$ and four moments:
  – Unconditional equity premium,
  – Unconditional variance premium,
  – Unconditional risk-free return level,
  – Covariance between $VIX^2(t)$ and $R_f(t+1)$. 