Nonparametric Option-Implied Volatility

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November 10, 2017

Outline

- Setup
- Close-to-Money Options as estimates of Volatility
- Characteristic Function based Volatility from Options
- Truncated Volatility from Options
- Feasible CLT
- Empirical Application

Setup

The underlying process is X and the dynamics for $x = \log X$ is given by:

$$x_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} x \widetilde{\mu}(ds, dx),$$

where

- W_t is a Brownian motion
- μ controls jumps
- all quantities are with respect to $\mathbb{Q}$

Our interest: nonparametric inference for σ_t from options.

Setup

We use short-dated options on X at time t, which expire at $t + T$:

$$O_T(k) = \begin{cases} \mathbb{E}_t^{\mathbb{Q}}(e^k - e^{x_{t+T}})^+, & \text{if } k \leq \ln F_T, \\ \mathbb{E}_t^{\mathbb{Q}}(e^{x_{t+T}} - e^k)^+, & \text{if } k > \ln F_T, \end{cases}$$

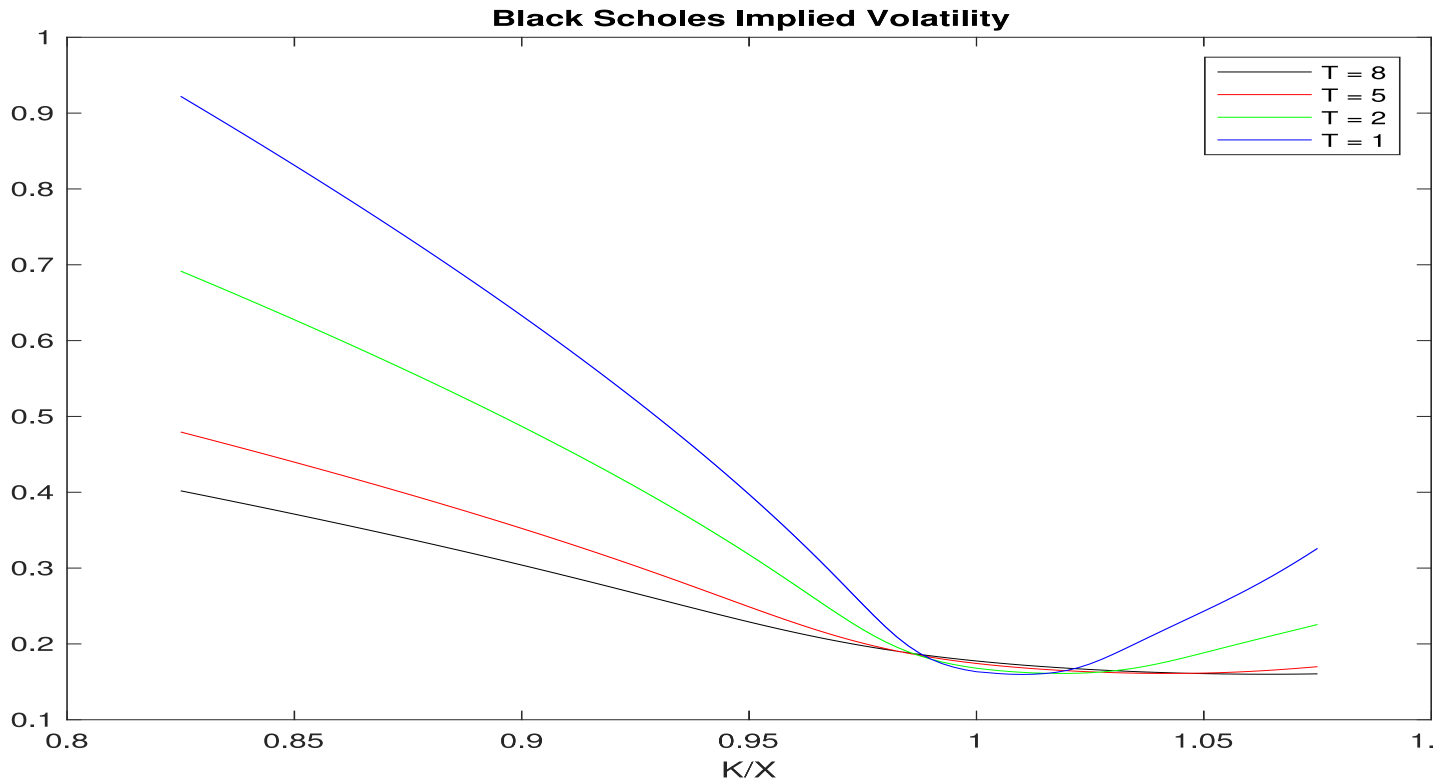
where F_T is the price at time t of a forward contract which expires at time $t + T$.

Setup

Option prices shrink with $T \downarrow 0$:

- $O_T(k) = O(\sqrt{T})$ for $|k| << \sqrt{T}$
- $O_T(k) = O(T)$ for fixed or asymptotically increasing $|k|$

Close-to-Maturity Option Convergence



Close-to-Money Option Expansion

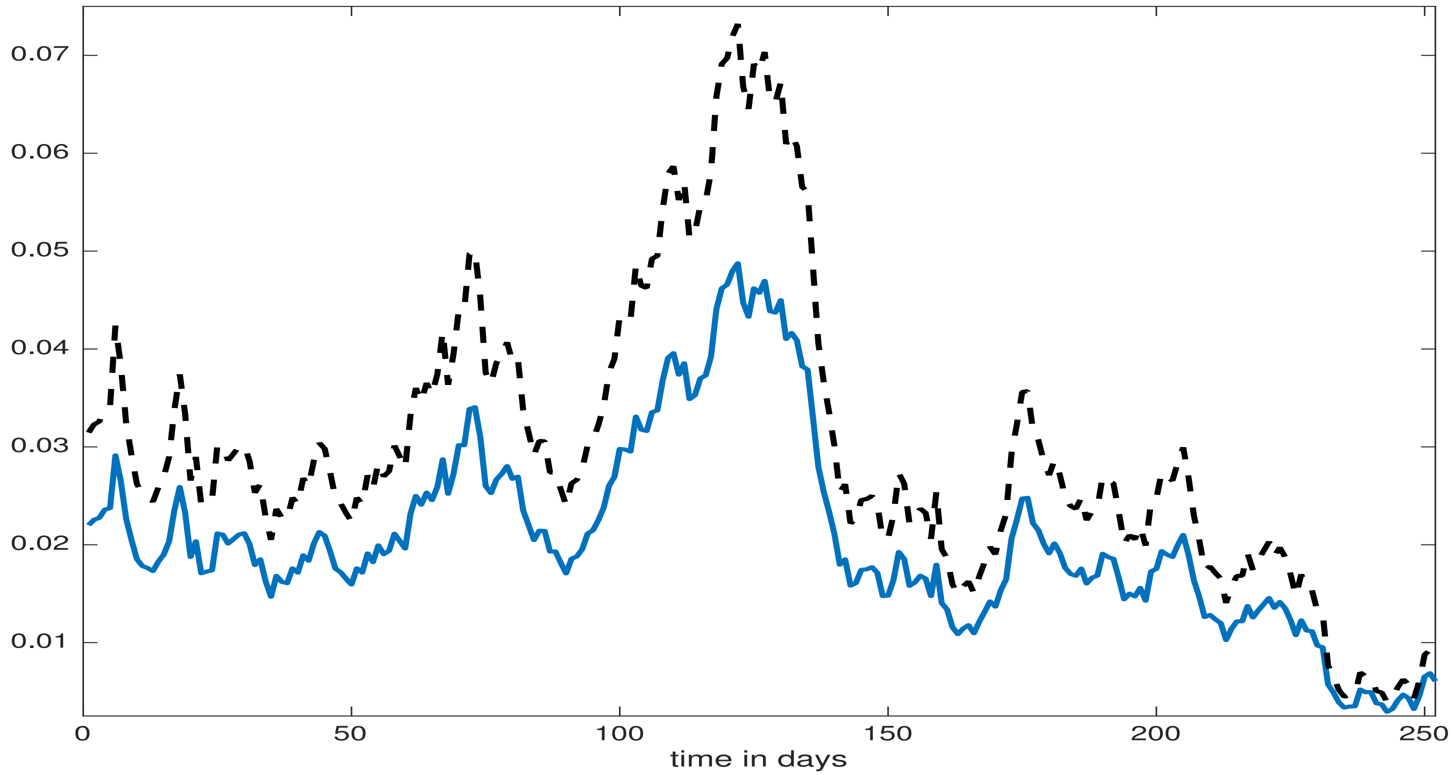
We have

$$O_T(k) = f\left(\frac{k_T}{\sqrt{T}\sigma_t}\right)\sqrt{T}\sigma_t - |e^{k_T} - 1|\Phi\left(-\frac{|k_T|}{\sqrt{T}\sigma_t}\right) + O_p(T),$$

where

- f and Φ are the pdf and cdf of standard normal,
- k_T is deterministic sequence with $k_T/\sqrt{T} = O_p(1)$,
- the expansion works in presence of jumps in X.

Close-to-Money Option Expansion



Option Portfolios

ATM options contain nontrivial bias due to the jumps.

We will try an alternative strategy by using portfolios of options with different strikes.

Following Carr and Madan (2001):

$$\mathbb{E}_t^{\mathbb{Q}}(f(X_T)) = f(F) + \int_{-\infty}^{\infty} f^{''}(e^k) O_T(k) e^k dk,$$

for any $f \in C^2$ and where F is the futures price at time t with expiration at $t + T$.

CF-Based Volatility

We can thus span:

$$\mathbb{E}_t^{\mathbb{Q}}\left(e^{iu(x_{t+T}-x_t)}\right) = 1 - (u^2 + iu)\int_{-\infty}^{\infty} e^{(iu-1)k - iux_t} O_T(k)dk, \quad u \in \mathbb{R}.$$

In the Lévy case:

$$\frac{1}{T}\log\left(\mathbb{E}_t^{\mathbb{Q}}\left(e^{iu(x_{t+T}-x_t)}\right)\right) = iua_t - \frac{u^2}{2}\sigma_t^2 + \int_{\mathbb{R}}(e^{iux} - 1 - iux)\nu_t(dx).$$

CF-Based Volatility

Therefore

$$\sigma_t^2 = -\frac{2}{Tu^2}\Re\left(\log\left(1-(u^2+iu)\int_{-\infty}^{\infty} e^{(iu-1)k-iux_t}O_T(k)dk\right)\right)$$
$$-\frac{2}{u^2}\int_{\mathbb{R}}(1-\cos(ux))\nu_t(dx),$$

and

$$\int_{\mathbb{R}}(1-\cos(ux))\nu_t(dx) \le C_t.$$

CF-Based Volatility

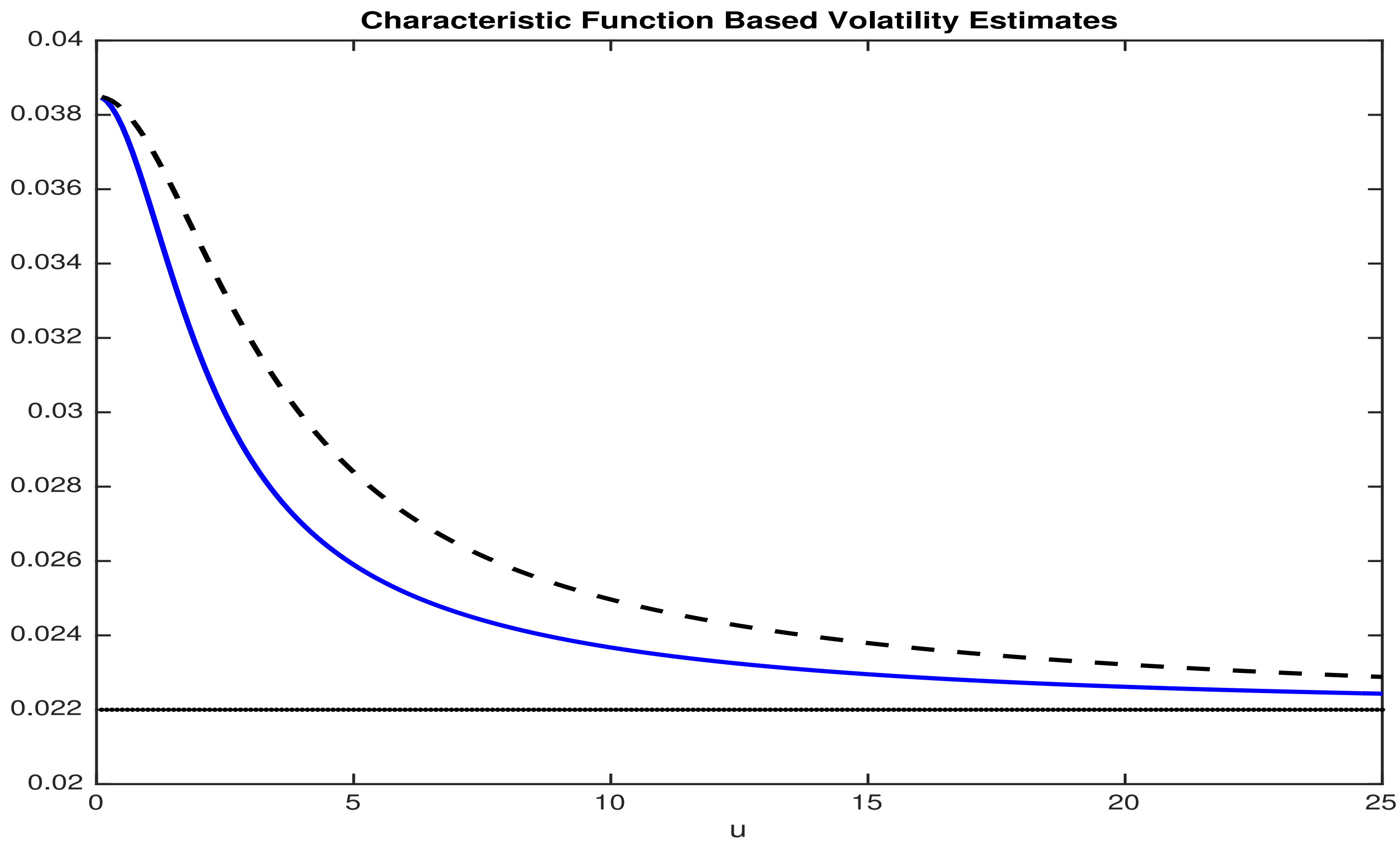
If we set

$$u_T \asymp \frac{1}{\sqrt{T}},$$

then

$$\sigma_t^2 = -\frac{2}{T u_T^2} \Re \left(\log \left(1 - (u_T^2 + i u_T) \int_{-\infty}^{\infty} e^{(i u_T - 1)k - i u_T x_t} O_T(k) dk \right) \right) + O\left(T\right).$$

CF-Based Volatility



CF-Based Volatility

In empirically realistic settings we have error due to

- presence of jumps in X
- finite number of options over a discrete grid of strikes
- observation error
- time-variation in characteristic triplet

We derive the order of magnitude of CF-based volatility estimator.

Observation Scheme

The available options are at time t, expiring at $t+T$, and having log-strikes given by:

$$\underline{k} \equiv k_1 < k_2 < \cdots k_N \equiv \overline{k},$$

with the corresponding strikes given by

$$\underline{K} \equiv K_1 < K_2 < \cdots K_N \equiv \overline{K}.$$

We denote

$$\Delta_i = k_i - k_{i-1}, \quad \text{for } i = 2, \ldots., N,$$

and assume for $\eta \in (0,1)$ a positive constant and deterministic $\overline{\Delta} \to 0$:

$$\eta\overline{\Delta} \leq \inf_{i=2,\ldots,N} \Delta_i \leq \sup_{i=2,\ldots,N} \Delta_i \leq \overline{\Delta}.$$

Observation Scheme

Instead of observing $O_T(k_i)$, we observe:

$$\widehat{O}_T(k_i) = O_T(k_i) + \epsilon_i.$$

We assume

$$\begin{cases} \mathbb{E}\left(\epsilon_i \middle| \mathcal{F}^{(0)}\right) = 0, \\ \epsilon_i \perp \epsilon_j, \quad \text{conditionally on } \mathcal{F}^{(0)}, \\ \mathbb{E}\left(\epsilon_i^2 \middle| \mathcal{F}^{(0)}\right) = O_T(k_i)^2 \sigma_{t,i}^2, \end{cases}$$

where $\sup_{i=1,\ldots,N} \sigma_{t,i}^2 = O_p(1)$.

CF-Based Volatility

The estimate of the conditional characteristic function is:

$$\widehat{f}_{t,T}(u) = 1 - (u^2 + iu) \sum_{j=2}^{N} e^{(iu-1)k_{j-1} - iux_t} \widehat{O}_T(k_{j-1}) \Delta_j, \quad u \in \mathbb{R},$$

We denote

$$\widehat{R}_{t,T}(u) = -\Re\left(\ln\left(\widehat{f}_{t,T}(u) \vee T\right)\right),$$

and then define

$$\widehat{V}_{t,T}(u) = \frac{2}{Tu^2} \widehat{R}_{t,T}(u).$$

CF-Based Volatility: Rate of Convergence

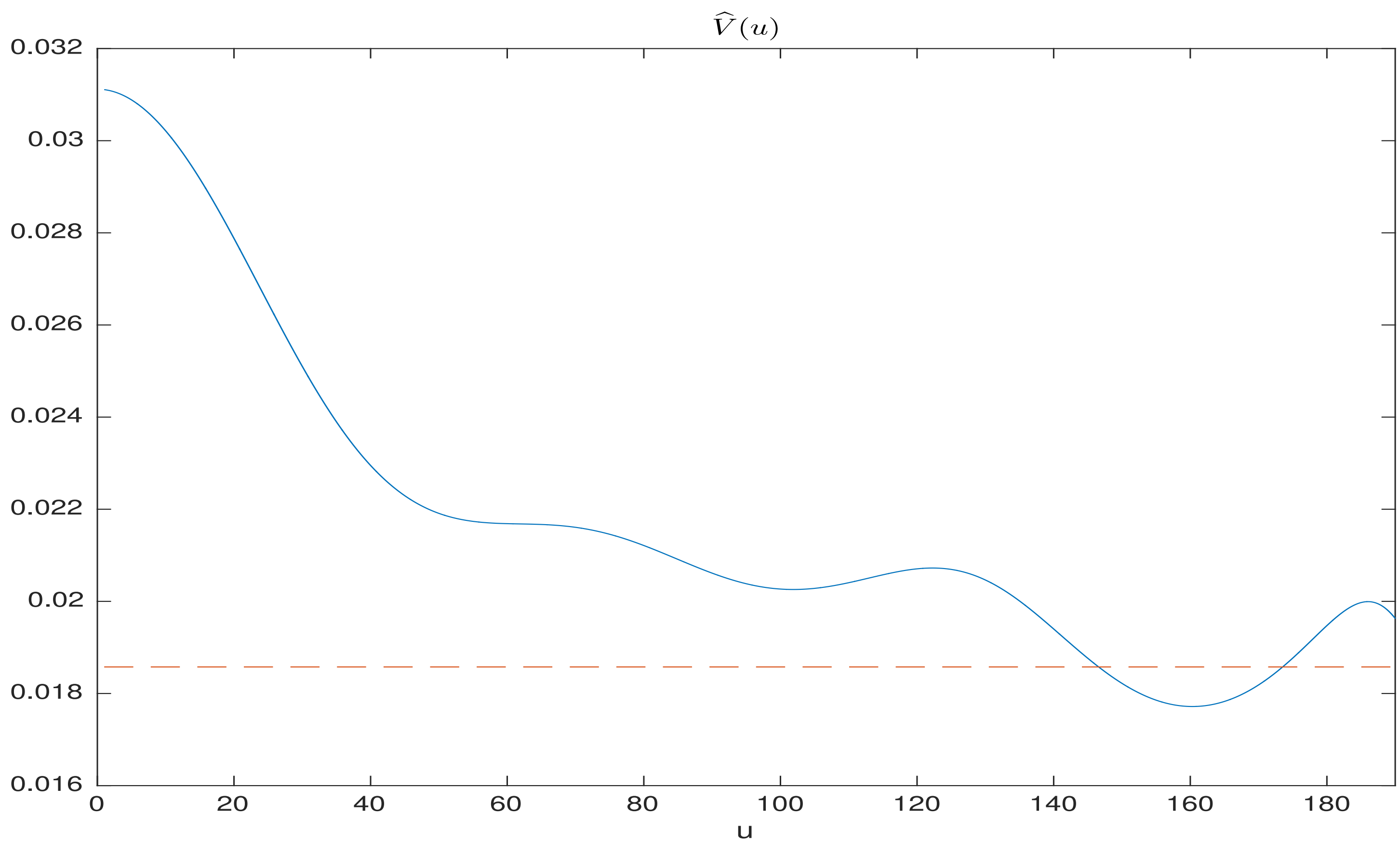
Theorem 1. *Suppose certain assumptions hold and in addition* $\overline{\Delta} \asymp T^{\alpha}$, $\overline{K} \asymp T^{-\beta}$, $\underline{K} \asymp T^{\gamma}$ *for some* $\alpha > 0$, $\beta > 0$ *and* $\gamma > 0$. *Let* $\{u_T\}_T$ *be an* $\mathcal{F}_t^{(0)}$*-adapted sequence such that*

$$u_T^2 T \xrightarrow{a.s.} \overline{u}, \quad \text{where } \overline{u} \text{ is a finite nonnegative random variable.}$$

Then, we have

$$\widehat{V}_{t,T}(u_T) - V_t = O_p\left(T^{1-\frac{r}{2}} \bigvee \frac{\sqrt{\overline{\Delta}}}{T^{1/4}} \bigvee e^{-2(|\underline{k}|\vee\overline{k})}\right).$$

CF-Based Volatility



CF-Based Volatility: Adaptive Estimation

Theoretically, any choice of $u_T \asymp 1/\sqrt{T}$ will work

In practice the choice of u_T is critical

From the characteristic function, the quantity that matters is: $u_T^2 \times \sigma_t^2 \times T$

We set u_T adaptively using a preliminary Truncation Volatility Estimator of σ_t

Truncated Volatility

We look at $f_\eta(x) = e^{-\eta x^2} x^2$ for $\eta > 0$.

We have

$$\begin{cases} f_\eta(x) \ \sim x^2, & \text{for } |x| \sim 0, \\ f_\eta(x) \le \frac{1}{\eta}, & \text{for } |x| > \frac{1}{\sqrt{\eta}}. \end{cases}$$

Therefore with $\eta_T \to \infty$, $f_{\eta_T}(x)$ can be used to separate volatility from jumps:

$$\left| \frac{1}{T} \mathbb{E}_t^{\mathbb{Q}} \left(f_{\eta_T}(\widetilde{x}_{t+T} - \widetilde{x}_t) \right) - \sigma_t^2 \right| = O_p \left(\sqrt{T} \vee \eta_T T \bigvee \frac{1}{\sqrt{\eta_T}} \right).$$

Truncated Volatility

Therefore, we look at

$$\int_{-\infty}^{\infty} h_\eta(k) O_T(k) dk,$$

where

$$h_\eta(k) = e^{-k-\eta(k-x_t)^2}\big[4\eta^2(k-x_t)^4 + 2 - 10\eta(k-x_t)^2 + 2\eta(k-x_t)^3 - 2(k-x_t)\big].$$

We have

$$\begin{cases} \int_{-\infty}^{\infty} h_0(k) O_T(k) dk = \sigma_t^2 + \int_{\mathbb{R}} x^2 \nu_t(dx) + o_p(1), \\ \int_{-\infty}^{\infty} h_{\eta_T}(k) O_T(k) dk = \sigma_t^2 + o_p(1), \quad \text{for } \eta_T \to \infty. \end{cases}$$

Truncated Volatility

The option-based Truncated Volatility estimator is defined by:

$$\widehat{TV}_{t,T}(\eta) = \frac{1}{T}\sum_{j=2}^{N} h_\eta(k_{j-1})\widehat{O}_T(k_{j-1})\Delta_j, \quad \eta \geq 0.$$

The total volatility estimator is:

$$\widehat{QV}_{t,T} \equiv \widehat{TV}_{t,T}(0).$$

We set the cutoff level adaptively at

$$\widehat{\eta}_T = \frac{\overline{\eta}_T}{T}\frac{1}{\widehat{QV}_{t,T}},$$

for some deterministic sequence $\overline{\eta}_T$ that depends only on T.

Truncated and Total Volatility: Consistency

Theorem 2. *Suppose certain assumptions hold and in addition* $\overline{\Delta} \asymp T^{\alpha}$, $\overline{K} \asymp T^{-\beta}$, $\underline{K} \asymp T^{\gamma}$ *for some* $\alpha > 0$, $\beta > 0$ *and* $\gamma > 0$. *If* $\alpha > \frac{1}{2}$, *we have*

$$\widehat{QV}_{t,T} \xrightarrow{\mathbb{P}} QV_{t,T}.$$

Suppose in addition that for $\overline{\eta}_T$:

$$\overline{\eta}_T \to 0 \text{ and } \frac{\overline{\eta}_T}{T} \to \infty.$$

Then, we also have

$$\widehat{TV}_{t,T}(\widehat{\eta}_T) \xrightarrow{\mathbb{P}} V_t.$$

Adaptive CF-Based Volatility

The adaptive choice for the characteristic exponent is given by

$$\widehat{u}_T = \frac{\overline{u}}{\sqrt{T}} \frac{1}{\sqrt{\widehat{TV}_{t,T}(\widehat{\eta}_T)}},$$

where $\overline{u}$ is some positive constant.

We further denote with $\widehat{Avar}(\widehat{V}_{t,T}(u))$ an estimate of the asymptotic variance based on

$$\widehat{\epsilon}_j = \widehat{O}_T(k_j) - \frac{1}{2}\left(\widehat{O}_T(k_{j-1}) + \widehat{O}_T(k_{j+1})\right), \quad \text{for } j = 2, \ldots, N-1.$$

CF-Based Volatility: CLT

Theorem 3. *Suppose certain assumptions hold and in addition* $\overline{\Delta} \asymp T^{\alpha}$, $\overline{K} \asymp T^{-\beta}$, $\underline{K} \asymp T^{\gamma}$ *for some* $\alpha > 0$, $\beta > 0$ *and* $\gamma > 0$. *If*

$$\frac{1}{2} < \alpha < \left(\frac{5}{2} - r\right) \bigwedge \left(\frac{1}{2} + 4(\beta \wedge \gamma)\right),$$

then

$$\frac{\widehat{V}_{t,T}(\widehat{u}_T) - V_t}{\sqrt{\widehat{Avar}(\widehat{V}_{t,T}(\widehat{u}_T))}} \xrightarrow{\mathcal{L}} N(0,1).$$

Empirical Application

With A Little Help from Yang Zhang

Empirical Application

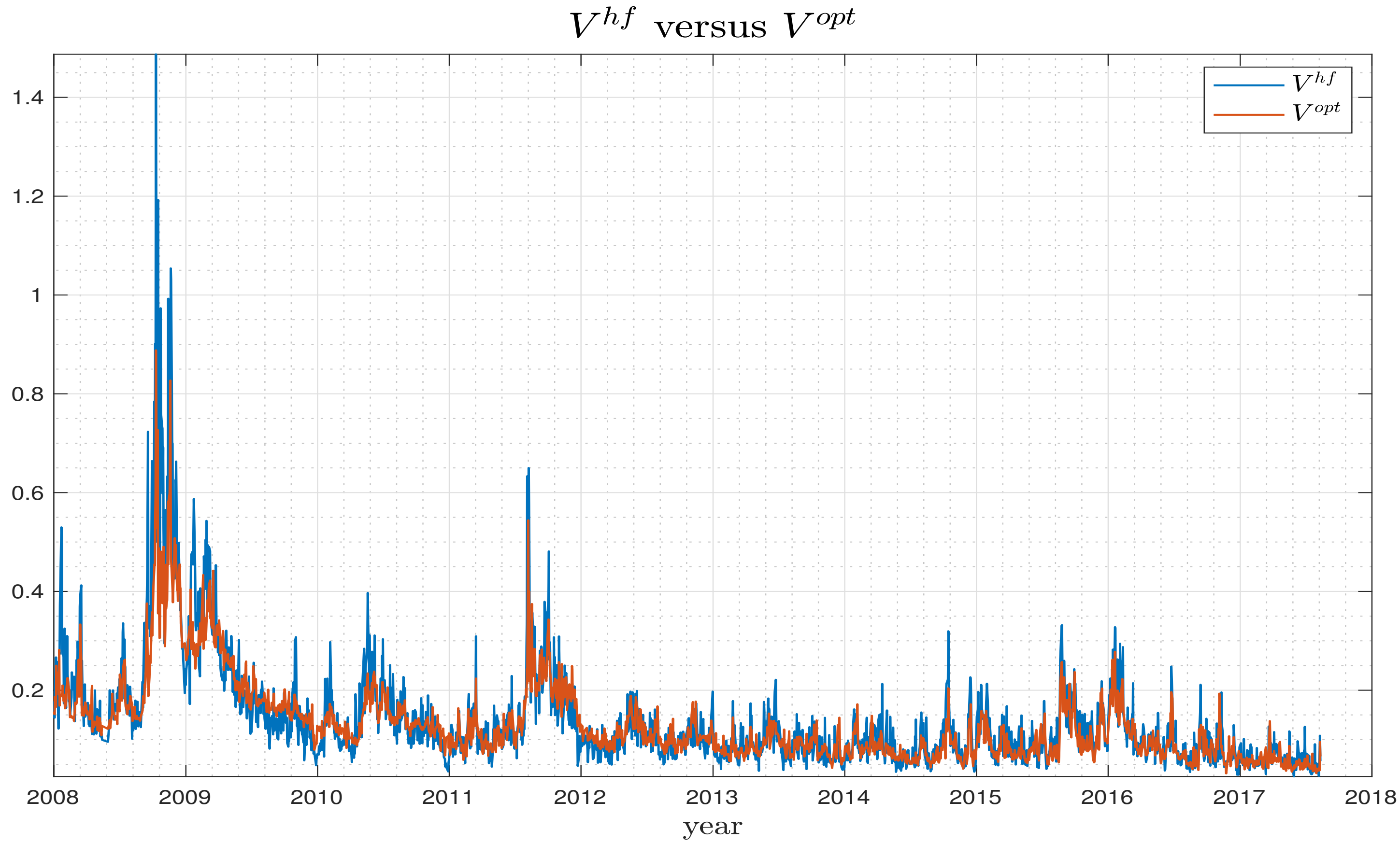
Option Data:

- SPX short maturity option data at market close
- period: 01/2008 - 08/2017 (weeklies start from 01/2011)
- maturity: 2 to 5 business days (based on weeklies)
- median size of option cross-section: 59 OTM options (based on weeklies)

HF Data:

- frequency: 5-minutes during work hours
- local window: trading day

Option vs HF Volatility Estimates



Empirical Application

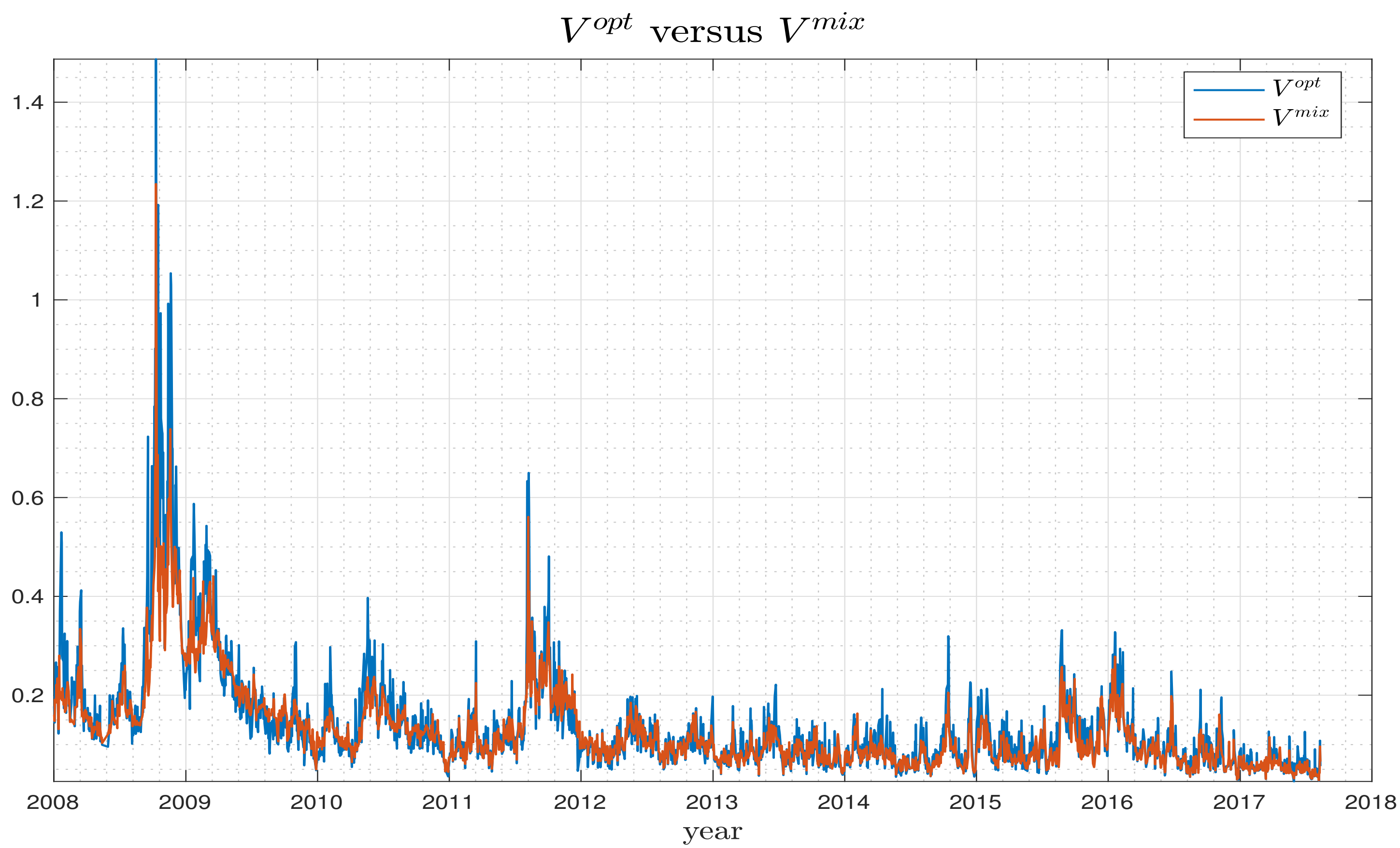
Consider optimal estimator:

$$V_t^{mix} = \omega_t \times V_t^{opt} + (1 - \omega_t) \times V_t^{hf},$$

where $\omega_t \in [0, 1]$ is optimal weight determined by asym. variance of the two estimators.

Median value of ω is: 0.85

HF vs Combined Volatility Estimates



Empirical Application

Gains for forecasting:

$$RV_{t+1} = \alpha_0 + \alpha_1 X_t + \epsilon_{t+1},$$

where X_t is a volatility predictor from the list:

- RV_t
- V_t^{opt}
- V_t^{hf}
- V_t^{mix}
- QRV_t

Forecast Performance Relative to Benchmark RV Forecast Model

Table 1: MSE

	Predictor			
	V_t^{opt}	V_t^{hf}	V_t^{mix}	QRV_t
Rolling Window	0.7445	1.0789	0.6886	0.6868
Increasing Window	0.6607	1.1438	0.6216	0.5600

Table 2: QLIKE

	Predictor			
	V_t^{opt}	V_t^{hf}	V_t^{mix}	QRV_t
Rolling Window	0.9828	1.0439	0.8560	0.9748
Increasing Window	0.9189	1.0618	0.8088	0.8826

Conclusion

- We propose nonparametric option-based volatility estimates
- The estimates are based on option portfolios of short-maturity options with different strikes
- Characteristic-based Option Portfolio for high frequencies separates volatility from jumps
- Tuning parameter selected from Option-based Truncated Volatility
- Empirical Application shows efficiency gains over HF Volatility Estimates